Bernoulli's necklace and the quadrisection of convex polygons.

Carl Eberhart and Fred Halpern

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Abstract

Given an arbitrary polygon $\mathcal{P}$ a **secant** XY (a segment with endpoints X, Y on the boundary of $\mathcal{P}$) is called a **bisector** if it divides $\mathcal{P}$ into 2 equal areas. Note each secant of $\mathcal{P}$ is parallel to exactly one bisector of $\mathcal{P}$. An orthogonal pair of bisectors of $\mathcal{P}$ is said to **quadrisect** $\mathcal{P}$ if they divide $\mathcal{P}$ into 4 equal areas.

In this paper, we use elementary vector algebra to develop an algorithm for finding all the quadrisections of an arbitrary convex polygon, establish some theorems about the quadrisection of certain classes of convex polygons.

1 Initial analysis and terminology

1.1 Background

Throughout, $\mathcal{P}$ will denote a convex polygon with vertices $P_0, P_1, \cdots, P_{n-1}$ labeled in counter clockwise order. A secant XY of $\mathcal{P}$ is a **bisector** (of $\mathcal{P}$) if XY separates $\mathcal{P}$ into two polygons of equal area. If XY is a bisector, then X and Y are called **antipodes**; we write $Y = X^*$ and $X = Y^*$. Two perpendicular bisectors form a **quadrisection** of $\mathcal{P}$ if together they separate $\mathcal{P}$ into four polygons of equal area.

Jacob Bernoulli [1] and Leonhard Euler [2] published solutions to the problem of constructing a quadrisection of an arbitrary triangle. Euler showed in particular that every scalene triangle has a quadrisection with one endpoint on the middle leg.

In [3], the first author proved a theorem describing **all quadrisections** of a triangle. It is shown there that the vast majority of triangles have exactly one quadrisection, although triangles sufficiently close in the Hausdorff metric[1] to an equilateral triangle have three quadrisections. The triangles that have exactly two quadrisections were seen to form an arc in the space of triangles separating the triangles with three quadrisections from those with only one quadrisection.

In light of this result, it is natural to look at the more general problem of finding the quadrisections of other types of plane figures. For example, it is not difficult to show that virtually any plane region with positive Lesbegue measure has at least one quadrisection. This is a consequence of the fact that as a line sweeps across the region, the area of the region on a fixed side of the line changes continuously.

In this paper, we develop an algorithm for computing all quadrisections of an arbitrary convex polygon, and use it to investigate how the quadrisections change as the polygon changes, with the ultimate aim of proving the conjecture stated in [3]: *A convex polygon with* $2n + 1$ *sides has at most* $2n + 1$ *quadrisections.* Although we have not settled this, we are able to provide a complete description of all quadrisections of the regular polygons (Section 5).

We have coded the algorithm in sage using an account on `https://cloud.sagemath.com`, and used it in the construction of several *sagelets* (my term) designed which we used to carry out the investigation. These sagelets can be found at `https://www.ms.uky.edu/~carl/sagelets`.

At this point, in order to implement our approach computationally, we will recall some elementary facts from analytic geometry and linear algebra.

1.2 More Terminology

[1]The Hausdorff metric measures how close two polygons $\mathcal{P}$ and $\mathcal{Q}$ are to each other. It is the smallest ϵ such that each vertex of $\mathcal{P}$ lies within ϵ of some vertex of the $\mathcal{Q}$ and each point of $\mathcal{Q}$ lies within ϵ of some point of $\mathcal{P}$. See pp. 280,281 [4]

It is an easy continuity argument to show that for each angle $\phi \in [0, \pi)$, there is exactly one bisector $BS(\phi) = X(\phi)Y(\phi)$ of $\mathcal{P}$, where $X(\phi)$, $(Y(\phi))$ is the endpoint of $BS(\phi)$ which comes first (second) in the counterclockwise order from P_0 a We can assume that $X(0) = P_0$. These two functions are clearly continuous, and in fact as we see later, they are piecewise differentiable.
For
each point $X = X(\phi)$ on the boundary of $\mathcal{P}$, let $X^{\perp} = X(\phi + \pi/2)$ and note that $X^{\perp}$ is the first endpoint (counterclockwise from X) of the unique bisector of $\mathcal{P}$ perpendicular to XX^*.
We call $X^{\perp}$ the **right perp of** X, so $X^{\perp *}$ would be the **left perp** of X. We call this pair of bisectors $Cross(X) = (XX^*, X^{\perp}X^{\perp *})$ the **cross of** X **in** $\mathcal{P}$, and we call the intersection point $Bp(X)$ of the cross the **Bernoulli point of** X **in** $\mathcal{P}$.
As ϕ moves from 0 to $\pi/2$, $X = X(\phi)$ moves counterclockwise along the boundary of $\mathcal{P}$ from P_0 to $P_0^{\perp}$, we will see that $Bp(X)$ traces out a piecewise smooth loop which we call **Bernoulli's necklace**.

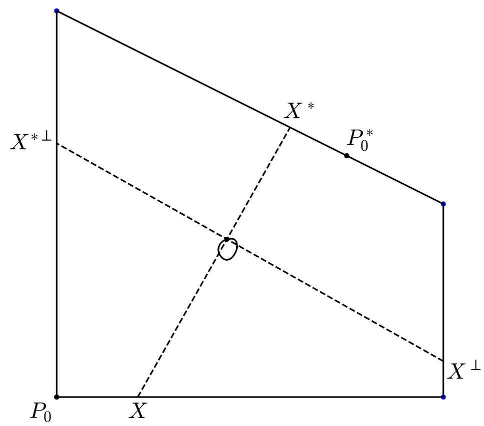


Figure 1

Quadrisections are crosses, and their intersection points, which we call **beads**, lie on Bernoulli's necklace.
In Figure 1, a quadrangle with a typical cross is illustrated, together with Bernoulli's necklace. Note this polygon has only the one quadrisection as evidenced by the single Bernoulli bead.
For each $\phi \in [0, \pi/2]$, we have $X(\phi)$ in the boundary of $\mathcal{P}$ between P_0 and $P_0^{\perp}$, define $Quad(\phi)$ to be that portion of $\mathcal{P}$ in the 'quadrant' of $\mathcal{P}$ determined by $X(\phi)\, Bp(X(\phi)), X(\phi + \pi/2)$, and define $AR(\phi)$ to be the area of $Quad(\phi)$.
The problem of finding the quadrisections of $\mathcal{P}$ can be restated: Find the angles $\phi \in [0, \pi/2)$ which satisfy the equation $AR(\phi) = 0.25\text{Area}(\mathcal{P})$.
Trapezoids play a key role in our approach to the problem of finding quadrisections of $\mathcal{P}$, because as we shall see, $\mathcal{P}$ is covered by a sequence of trapezoids, and the 'area preserving' diagonals of a trapezoid have a nice parameterization.

2 Digression on trapezoids

In this section, $\mathcal{T}$ is a trapezoid with vertices A, B, C, D labeled in counterclockwise order with **bases** AD parallel with BC, **sides** AB and CD and **diagonals** AC and BD. Relative to this order, we will refer to AB as the **first side** and AC is the **first** diagonal of $\mathcal{T}$. Let E be the intersection of the first and second diagonals.
Then $\angle AEB = \theta$ is the **central angle of** $\mathcal{T}$, and $\angle CAB = \alpha$ and $\angle ACD = \beta$ are the **first** and **second diagonal angles** of $\mathcal{T}$. Note that $\mathcal{T}$ is completely determined up to isometry by the three angles $\theta,\ \alpha,\ \beta$ and the lengths of the first diagonal and first side.

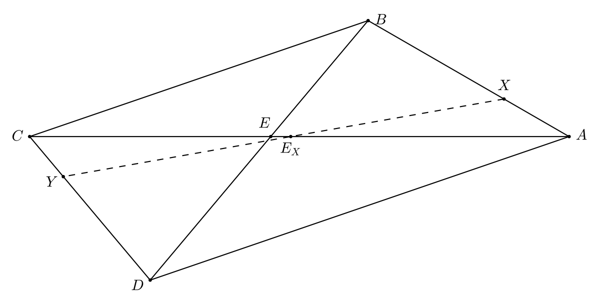


Figure 3

Lemma 1. *ΔABF and ΔFCD have the same area.*

Proof. The triangles ΔACD and ΔABD have the same base AD and the same height, the distance between the lines AD and BC, so they have the same area. But area ΔACD = area $\Delta AFD +$ area ΔFCD and areaΔABD = area $\Delta AFD +$ area ΔABF. Hence area ΔFCD = area ΔABF. □

By a **diagonal** of $\mathcal{T}$, we mean a segment XY with $X \in AB$ and $Y \in CD$ such that area $AXYD$ = area ABD. Thus BD and AC are diagonals.

Lemma 2. *If X_1Y_1 and X_2Y_2 are diagonals of $\mathcal{T}$, then X_1Y_2 and X_2Y_1 are parallel (and so $X_1X_2Y_1Y_2$ is a trapezoid).*

Proof. We can assume that X_1 is between A and X_2 and Y_2 is between D and Y_1. Hence area AX_1Y_1D = area AX_2Y_2D. Hence area $X_1X_2Y_2$ = area AX_1Y_1D − area AX_1X_2D = area AX_2Y_2D − area AX_1X_2D = area $X_1Y_1X_2$. Since these triangles have the same base X_1X_2 they have the same height and so X_2Y_1 is parallel with AD. ☐

Theorem 1. *For each angle ϕ between 0 and θ, there is a unique diagonal XY intersecting AC in the angle ϕ. Further, the quadrilateral $\square AXCY$ is a trapezoid with XC parallel with YA.*

Proof. Choose $X' \in AB$ so that $\angle X'FA$ has measure ϕ, and let Y' be the intersection of line $X'F$ with CD. If area $AX'Y'D \neq$ area ΔACD, then by continuity of the area function we can move $X'Y'$ parallel to (a unique) XY so that area $AXYD$ = area ΔACD. XC is parallel with YA by lemma 2. ☐

This theorem tells us that there is an arc of diagonals of $\mathcal{T}$ from AC to BD. We will see that in the movement of the diagonal XY from AC to BD, it rotates about its midpoint so that the area of the quadrilateral $AXYD$ remains unchanged. That

2.1 Parameterizing the diagonals of a trapezoid.

Now we will compute the diagonal XY as a function of the angle $\phi = t\,\theta$ for t in $[0, 1]$.
In order to simplify calculations, we use a translation and rotation to reposition $\mathcal{T}$ so that $A = (a, 0)$ and $C = (-a, 0)$, where $2\,a = |A - C|$. Then applying the reverse motions, we obtain a parameterization for XY.
Let X and Y be the unique points guaranteed by Theorem 1. Let $E = (s, 0)$ be the intersection of XY with AC. Since AC and XY are diagonals, AY and XC are parallel, and ΔAXE and ΔDEY have the same area, that is $AE\,XE\,\sin(t\,\theta) = DE\,EY\,\sin(\alpha)$. From the law of sines, we know $|X - E| = |A - E|\dfrac{\sin(\alpha)}{\sin(\alpha + t\theta)}$, and $|Y - E| = |D - E|\dfrac{\sin(\beta)}{\sin(\beta + t\theta)}$. Note that $|A - E| = a - s$, and $|D - E| = a + s$. Hence we have the endpoints X and Y and the length $|Y - X|$ of the diagonal as functions of s and t.
$X = E + (a - s)\dfrac{\sin(\alpha)}{\sin(\alpha + t\theta)}U_{t\theta}$, and $Y = E - (a + s)\dfrac{\sin(\beta)}{\sin(\beta + t\theta)}U_{t\theta}$, where $U_{t\theta} = (\cos(t\theta), \sin(t\theta))$.
$|X - Y| = (a - s)\dfrac{\sin(\alpha)}{\sin(\alpha + t\theta)} + (a + s)\dfrac{\sin(\beta)}{\sin(\beta + t\theta)}$.
We have a quadratic equation in s and t by equating the areas of ΔAXE and ΔDEY:

$$(a - s)^2\frac{\sin(\alpha)}{\sin(\alpha + t\theta)} = (a + s)^2\frac{\sin(\beta)}{\sin(\beta + t\theta)}$$

Note that s is between $-a$ and a and $\alpha + t\theta$ and $\beta + t\theta$ are both between 0 and π, hence we can equate the positive square root of both sides and solve the resulting linear equation in s for s to get s as a function of t:

$$\text{II}).\ s(t) = a\,\frac{\sqrt{\sin(\beta + t\theta)} - r\sqrt{\sin(\alpha + t\theta)}}{\sqrt{\sin(\beta + t\theta)} + r\sqrt{\sin(\alpha + t\theta)}},\ \text{where } r = \sqrt{\frac{\sin(\beta)}{\sin(\alpha)}}$$

The derivative of $s(t)$ simplifies to

$$\text{III}).\ s'(t) = a\theta\,r\left(\frac{\frac{1}{r(t)}\cos(\beta + t\theta) - r(t)\cos(\alpha + t\theta)}{(\sqrt{\sin(\beta + t\theta)} + r\sqrt{\sin(\alpha + t\theta)})^2}\right)\ \text{where } r(t) = \sqrt{\frac{\sin(\beta + t\theta)}{\sin(\alpha + t\theta)}}$$

.
These two formulas give us geometrical information about the motion of $E(t)$ on the diagonal AC of our trapezoid.

Theorem 2. *$E(t)$ starts at the midpoint of AC, and 'track' the midpoint $M(t)$ of $X(t)Y(t)$ while staying on AC depending on the relation of α to β. Thus*
1) $E(t)$ remains stationary at the midpoint of AC if $\alpha = \beta$,
2) $E(t)$ moves towards A as $M(t)$ heads toward B if $\alpha > \beta$, and
3) $E(t)$ moves towards C as $M(t)$ heads toward D if $\beta > \alpha$.

Proof. First $s(0) = 0$ always, so $F = E(0) = (0, 0)$ the midpoint of AC. Next, if $\alpha = \beta$, then the numerator of $s(t)$ is always 0, so $E(t) = (0, 0)$ for all $t \in [0, 1]$. This establishes 1). To establish 2) and 3), note that the sign of $s'(t)$ is the sign of the numerator $\frac{1}{r(t)}\cos(\beta + t\theta) - r(t)\cos(\alpha + t\theta)$. Set this to 0 and simplify to the equation $\cot(\beta + t\theta) = \cot(\alpha + t\theta)$. Now suppose t_0 is a solution for some $t_0 \in [0, 1]$. Since $0 < \beta + t\theta, \alpha + t\theta < \pi$ and cot is 1-1 and decreasing on $(0, \pi)$, we have $\beta + t_0\theta = \alpha + t_0\theta$. But then $\beta = \alpha$, in which case $s'(t) = 0$ for all $t \in [0, 1]$. Since $s'(t)$ is continuous, it follows that if $\alpha > \beta$, then $s'(t) > 0$ for all $t \in [0, 1]$, establishing 2), and if $\beta > \alpha$, then $s'(t) < 0$ for all $t \in [0, 1]$, establishing 3). □

In figure 5, $\alpha < \beta$ and $s' < 0$, so $E(t)$ moves to the left from $E(0)$, the midpoint of AC, towards C, ending at $F = E(1)$, the intersection of the main diagonals of $\mathcal{T}$.
Now that we have the length $|X - Y|$ of the the diagonal XY as a function of t we can compute it's derivative. Note: although we can compute this derivative directly from our formula for $|X - Y|$, we will derive a more useful form for our purpose. This leads in turn to a nicer compact form for the length of XY.

Theorem 3. *1)* $\frac{d\,|X - Y|}{dt} = -.5|X - Y|\theta\,(\cot(\alpha + t\theta) + \cot(\beta + t\theta))$.
2) $|X - Y| = |A - C|\sqrt{\frac{\sin(\alpha)\sin(\beta)}{\sin(\alpha + t\theta)\sin(\beta + t\theta)}}$.

Proof. Fix $t \in [0, 1]$ and let h be a number close to 0. We will assume $h > 0$. Then $X(t)X(t + h)Y(t)Y(t + h)$ is a trapezoid with diagonals $X(t)Y(t)$ and $X(t + h)Y(t + h)$. Let F be the intersection of the diagonals. Let $m = |X(t) - F|, k = |X(t + h) - F|, n = |Y(t) - F|$, and $l = |Y(t + h) - F|$. By the law of sines, we have $m = k\frac{\sin(\alpha + t\theta + h\theta)}{\sin(\alpha + t\theta)}$ and $n = l\frac{\sin(\beta + t\theta + h\theta)}{\sin(\beta + t\theta)}$
Now, $\frac{d\,|X - Y|}{dt} = \lim_{h\to 0}\frac{(k + l) - (m + n)}{h} = \lim_{h\to 0} -\frac{(m + n) - (k + l)}{h}$, so we need to simplify this.

$$
\begin{aligned}
-\frac{(n + m) - (k + l)}{h} &= -\frac{(l\frac{\sin(\beta + t\theta + h\theta)}{\sin(\beta + t\theta)} + k\frac{\sin(\alpha + t\theta + h\theta)}{\sin(\alpha + t\theta)}) - (k + l)}{h} \\
&= -\frac{(l(\frac{\sin(\beta + t\theta + h\theta)}{\sin(\beta + t\theta)} - 1) + k(\frac{\sin(\alpha + t\theta + h\theta)}{\sin(\alpha + t\theta)} - 1)}{h} \\
&= -\theta(l(\frac{\cos(h\theta) - 1}{h\theta} + \cot(\beta + t\theta)\frac{\sin(h\theta)}{h\theta}) + k(\frac{\cos(h\theta) - 1}{h\theta} + \cot(\alpha + t\theta)\frac{\sin(h\theta)}{h\theta}))
\end{aligned}
$$

Now as h tends to 0, l and k both tend to $.5|X - Y|$ (since the areas of $\Delta FX(t)X(t + h)$ and $\Delta FX(t)X(t + h)$ remain equal), $\frac{\cos(h\theta) - 1}{h\theta}$ tends to 0 and $\frac{\sin(h\theta)}{h\theta}$ tends to 1. This proves 1).

To prove 2), note that $\frac{\frac{d|X - Y|}{dt}}{|X - Y|} = -.5(\cot(\alpha + t\theta)\theta + \cot(\beta + t\theta)\theta)$. Integrate and solve for $|X - Y|$ to get $|X - Y| = \frac{K}{\sqrt{\sin(\alpha + t\theta)\sin(\beta + t\theta)}}$ for some constant K. At $t = 0$, K must be $2a\sqrt{\sin(\alpha)\sin(\beta)}$. Note $2a = |A - C|$ and 2) is established. □

With this form for the derivative of $|X - Y|$ we can easily determine when $|X - Y|$ is increasing, stationary, or decreasing.

Theorem 4. *Let* $t_{min} = \max(-\alpha/\theta, -\beta/\theta)$*,* $t_{max} = \min(\pi - \alpha/\theta, \pi - \beta/\theta)$*, and* $t_z = \dfrac{\pi - (\alpha + \beta)}{2\theta}$*. The If* $\alpha + \beta + 2\theta < \pi$ *then* $|X - Y|$ *is decreasing on* $[0, 1]$*, if* $t_s \in [0, 1]$*, then* $|X - Y|$ *is decreasing on* $[0, t_s]$ *and increasing on* $[t_s, 1]$ *and if* $\pi < \alpha + \beta$ *then* $|X - Y|$ *is increasing on* $[0, 1]$*.*

Proof. $\dfrac{d|X - Y|}{dt} = 0$ only when $\cot(\alpha + t\theta) + \cot(\beta + t\theta) = 0$. Since both $\alpha + t\theta$ and $\beta + t\theta$ lie in $(0, \pi)$ for all t, the only solution to $\cot(\alpha + t\theta) + \cot(\beta + t\theta) = 0$ is when $\dfrac{\alpha + t\theta + \beta + t\theta}{2} = \dfrac{\pi}{2}$, that is, $t = t_s = \dfrac{\pi - (\alpha + \beta)}{2\theta}$. If $0 <= t < t_s$, then $\alpha + t\theta < t_s\theta$ □

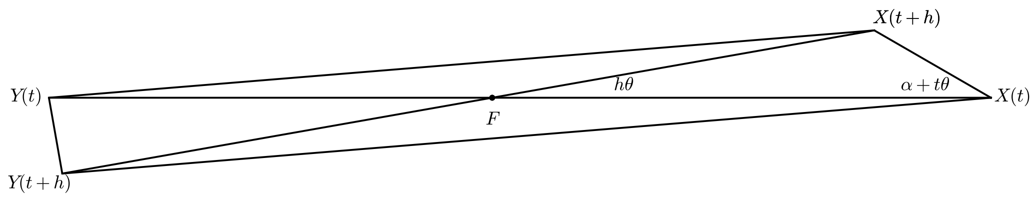


Figure 4

3 The covering of a polygon by orthogonal pairs of trapezoids

For each $i = 0, \cdots, n - 1$, let $C_i = \{P_i P_i^*, P_i^\perp P_i^{\perp *}\}$ be the cross determined by P_i. Note that $C_i = C_j$ if and only if $P_j \in \{P_i, P_i^*, P_i^\perp, P_i^{\perp *}\}$. So there are k_n, where $1 \le k_n \le n$, crosses altogether, yield $4\,k_n$ distinct endpoints. Label the endpoint of each cross which lies in $\partial\mathcal{P}$ between P_0 and $P_0^\perp$ in counterclockwise order: $P_0 = Q_0, \cdots, Q_{k_n - 1}$ and let $Q_{k_n} = P_0^\perp$.
Now, for $i = 0, \cdots, k_n - 1$ let $\mathcal{Q}_i$ be the quadrangle $Q_i Q_{i+1} Q_i^* Q_{i+1}^*$, and let $\mathcal{Q}_i^\perp$ be the quadrangle $Q_i^\perp Q_{i+1}^\perp Q_i^{*\perp} Q_{i+1}^{*\perp}$. Let E_i and $E_i^\perp$ be the intersection of the diagonals of $\mathcal{Q}_i$ and $\mathcal{Q}_i^\perp$ respectively.

Lemma 3. *For* $i = 0, \cdots, k_n - 1$*,* $\mathcal{Q}_i$ *is a trapezoid with* $Q_i Q_{i+1}^*$ *parallel with* $Q_{i+1} Q_i^*$*. Similarly,* $\mathcal{Q}_i^\perp$ *is a trapezoid with* $Q_i^\perp Q_{i+1}^{*\perp}$ *parallel with* $Q_{i+1}^\perp Q_i^{*\perp}$*.*

Proof. Since $Q_i Q_i^*, Q_{i+1} Q_{i+1}^*$ are bisectors of $\mathcal{P}$, the area of that part of $\mathcal{P}$ on the Q_{i+1}-side of $Q_i Q_i^*$ equals the area on the Q_i^*-side of Q_{i+1}, Q_{i+1}^*. Hence, the area of $\Delta Q_i Q_{i+1} Q_i^*$ equals the area of $\Delta Q_{i+1} Q_i^* Q_{i+1}^*$. Since these two triangles have the same base $Q_{i+1} Q_i^*$, they have the same height. Thus $Q_i Q_{i+1}^*$ is parallel with $Q_{i+1} Q_i^*$. □

3.1 Orthogonal Trapezoids

The trapezoids $\mathcal{Q}_i$ and $\mathcal{Q}_i^\perp$ are examples of what we call an orthogonal pair trapezoids. More generally, two trapezoids $\mathcal{T}_1 = A_1 B_1 C_1 D_1$ and $\mathcal{T}_2 = A_2 B_2 C_2 D_2$ are an **orthogonal pair of trapezoids** (or simply **orthogonal trapezoids**) provided the their intersection is a parallelogram, $A_1 C_1$ is orthogonal with $A_2 C_2$ and $B_1 D_1$ is orthogonal with $B_1 D_1$. We are assuming the vertices are labeled so that A_2 is on the B_1-side of $A_1 C_1$ or on $A_1 C_1$, in order to assure that the ordering $A_1 B_1 A_2 B_2 C_1 D_1 C_2 D_2$ is counterclockwise.
From the way $\mathcal{Q}_i$ and $\mathcal{Q}_i^\perp$ are defined, it is clear that they are orthogonal trapezoids. Let Ff (resp. Ss, Fs, Sf) be the intersection of the first and first (resp. second and second, first and second, second and first) diagonals of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.
The following theorem is important for our approach to the quadrisection problem.

Theorem 5. *If the trapezoids* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are orthogonal, then*

1. $\angle A_1 E_1 B_1 = \theta_1 = \angle A_2 E_2 B_2$*; that is their central angles are equal.*
2. $\angle A_1 Fs B_2 = \frac{\pi}{2} + \theta$*.*

Proof. 1. Let $_1$ be the line through Ff parallel to B_1D_1. Choose $L_1 \in {}_1$ on the B_1-side of A_1C_1. Then $\angle L_1FfA_1 = \angle B_1E_1A_1 = \theta_1$, and L_1 lies in the interior of the right angle $\angle A_1FfA_2$. Hence, $\pi/2 = \angle A_1FfA_2 = \angle A_1FfL_1 + \angle L_1FfA_2 = \angle A_1FfL_1 + \theta_1$.
Now let $_2$ be the line through Ff parallel to B_2D_2 and choose L_1 on the B_2-side of A_2C_2. Then $\angle L_2FfA_2 = \angle B_2E_2A_2 = \theta_2$, and A_2 lies in the interior of $\angle L_1FfL_2$, which is a right angle since $_2$ is parallel with B_2D_2. Hence and $\pi/2 = \angle L_1FfL_2 = \angle L_1FfA_2 + \angle A_2FfL_2 = \angle L_1FfA_2 + \theta_2$. It follows that $\theta_1 = \theta_2$.
2. Let $_1$ be the line through Fs parallel to B_1D_1. Choose $L_1 \in {}_1$ on the B_1-side of A_1C_1. Then L_1 lies in the interior of $\angle A_1FsB_2$, hence $\angle A_1FsB_2 = \angle A_1FsL_1 + \angle L_1FsB_2 = \theta + \pi/2$ □

4 Further results on orthogonal trapezoids.

In this section, $\mathcal{T}_1 = A_1B_1C_1D_1$ and $\mathcal{T}_2 = A_2B_2C_2D_2$ are assumed to be orthogonal trapezoids with first diagonals A_1C_1 and A_2C_2. Denote $\angle C_iA_iB_i$ and $\angle A_iC_iD_i$ by α_i and β_i respectively for $i \in 1, 2$. Since the angle parameterization of a trapezoid depends only on the trapezoid each of our trapezoids comes equipped with diagonals $X_1(t)Y_1(t)$ and $X_2(t)Y_2(t)$ angle parameterized. By definition, $A_1C_1 = X_1(0)Y_1(0)$ is orthogonal with $A_2C_2 = X_2(0)Y_2(0)$ and $B_1D_1 = X_1(1)Y_1(1)$ is orthogonal with $B_2D_2 = X_2(1)Y_2(1)$. In fact, as $X_1(t)Y_1(t)$ and $X_2(t)Y_2(t)$ rotate at a constant rate, they will stay orthogonal. We prove this in the next theorem.

Theorem
6. *For each $t \in [0, 1]$, $X_1(t)Y_1(t)$ and $X_2(t)Y_2(t)$ are orthogonal.*

Proof. By Theorem 5, the central angles θ_1 and θ_2 are equal. Hence $t\theta_1 = t\theta_2$ for each $t \in [0, 1]$. So $X_1(t)Y_1(t)$ makes the same angle with A_1C_1 that $X_2(t)Y_2(t)$ makes the same angle with A_2C_2. Since A_1C_1 is orthogonal with A_2C_2, this means that $X_1(t)Y_1(t)$ is orthogonal with $X_2(t)Y_2(t)$. □

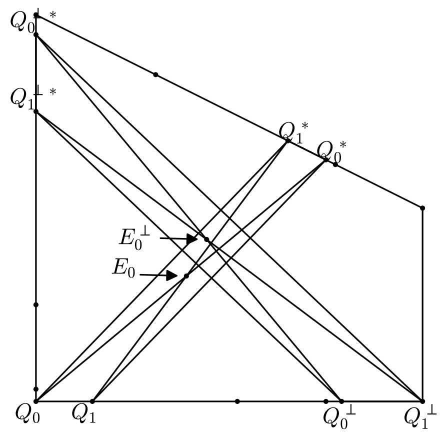


Figure 2

For each $t \in [0, 1]$, define $Bp(t)$ to be the intersection of the diagonals $X_1(t)Y_1(t)$ and $X_2(t)Y_2(t)$. Then $Bp(t)$ traces out a smooth path from $Ff = Bp(0)$ to $Ss = Bp(1)$. If $\mathcal{T}_1 = \mathcal{Q}_i$ and $\mathcal{T}_2 = \mathcal{Q}_i^\perp$ for some i, then this path is a portion of what we have called the Bernoulli necklace of $\mathcal{P}$.
Define the **midpoint functions** $M_1(t) = .5(X_1(t) + Y_1(t))$ and $M_2(t) = .5(X_2(t) + Y_2(t))$. As we have seen, as t goes from 0 to 1, the diagonal $X_i(t)Y)i(t)$ rotates counterclockwise about $M_i(t)$ for $i = 1, 2$
Define the four **Area functions for the orthogonal pair $\mathcal{T}_1$ and $\mathcal{T}_2$** in the following manner:
For $t \in [0, 1]$, define $Ar_1(t)$ to be the area of that part of the pair cut off by the right angle $X_1(t)Bp(t)X_2(t)$. See the shaded area in Figure 4. Also define $Ar_2(t)$, (respectively $Ar_3(t)$, $Ar_4(t)$) to be the area of that part of the pair cut off by the right angle $X_2(t)Bp(t)Y_1(t)$ (respectively, $Y_1(t)Bp(t)Y_2(t)$,$Y_2(t)Bp(t)X_1(t)$).
A very useful fact about these four functions is that once we know one of them, we know the other three, as the next theorem shows.

Theorem 7. *There are constants s_i, $i = 2, 3, 4$, such that $Ar_2(t) = s_2 - Ar_1(t)$, $Ar_3(t) = s_3 + Ar_1(t)$, and $Ar_4(t) = s_4 - Ar_1(t)$ for all $t \in [0, 1]$. Hence Ar_3 and Ar_1 have the same derivative, and Ar_2 and Ar_4 have the same derivative, namely $-Ar_1'$.*

Proof. Since $X_1(t)Y_1(t)$ is a diagonal of $\mathcal{T}_1$, the area of the quadrangle $X_1(t)B_1C_1Y_1(t)$ is constant over $[0, 1]$. But $Ar_1(t) + Ar_2(t)$ is the area of this quadrangle plus the area of the quadrangle $Q_1A_2B_2Q_2$ where Q_1 is the intersection of B_1C_1 with A_2D_2, and Q_2 is the intersection of B_1C_1 with B_2C_2. Hence $Ar_1(t) + Ar_2(t)$ is a constant say h_1. In the same way $Ar_2(t) + Ar_3(t)$ is a constant h_2, $Ar_3(t) + Ar_4(t)$ is a constant h_3, and $Ar_4(t) + Ar_1(t)$ is a constant h_4. So $s_2 = h_1$, $s_3 = h_2 - h_1$, and $s_4 = h_3$. □

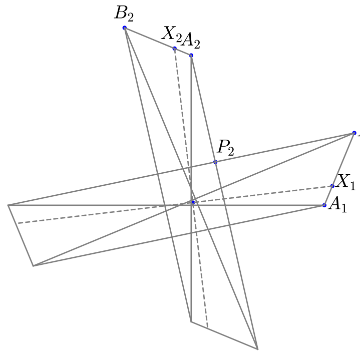


Figure 5.

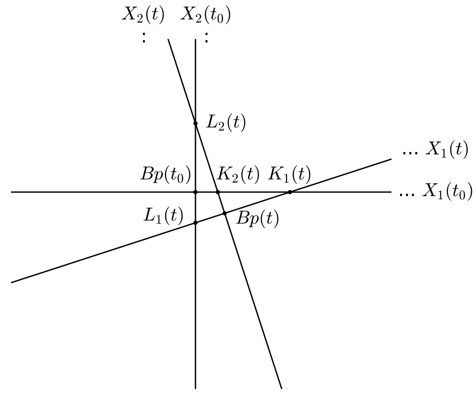


Figure 6.

Our main interest in defining Ar_1 is when the pair $\mathcal{T}_1$ and $\mathcal{T}_2$ is one of the covering pairs $\mathcal{Q}_i$ and $\mathcal{Q}_i^{\perp}$ of the polygon $\mathcal{P}$. For in that case, the area function $AR(\phi)$, defined in Section 1 differs from $Ar_1(t)$ by a constant amount, namely the area of the polygon $B_1 \cdots P_j \cdots A_2 K_1$, where the $P_j's$ (if any) lie between B_1 and A_2 on the boundary of $\mathcal{P}$.
Thus AR and Ar_1 have the same derivative. And we can calculate the derivative of Ar_1.

Theorem 8. *As a function of t,*
1). $Ar_1'(t) = .5\,\theta_1\,(|X_2(t) - Bp(t)||Y_2(t) - Bp(t)| - |X_1(t) - Bp(t)||Y_1(t) - Bp(t)|)$.
2). $Ar_1'(t) = .5\,\theta_1\,(|X_2(t) - Y_2(t)|^2 - |Md_2(t) - Bp(t)|^2 - (|X_1(t) - Y_1(t)|^2 + |Md_1(t) - Bp(t)|^2))$.
As a function of $\phi = t\theta_1$,
3) $Ar_1'(\phi) = .5\,(|X_2(\phi) - Bp(\phi)||Y_2(\phi) - Bp(\phi)| - |X_1(\phi) - Bp(\phi)||Y_1(\phi) - Bp(\phi)|)$
4) $Ar_1'(\phi) = .5\,(.25\,|X_2(\phi) - Y_2(\phi)|^2 - |Md_2(\phi) - Bp(\phi)|^2 - (.25\,|X_1(\phi) - Y(\phi)|^2 - |Md_1(\phi) - Bp(\phi)|^2))$

Proof. Fix $t_0 \in [0,1]$. For $i = 1,2$ and $t \in [0,1]$ with $t \neq t_0$, define $L_i(t)$ to be the intersection of $X_2(t_0)Y_2(t_0)$ and $X_i(t)Y_i(t)$, and define $K_i(t)$ to be the intersection of $X_1(t_0)Y_1(t_0)$ with $X_i(t)Y_i(t)$.
We note that $Bp(t_0)$, $E_1(t)$, and $K_1(t)$ lie in some order on the diagonal $X_1(t_0)Y_1(t_0)$ and $Bp(t)$, $E_2(t)$, and $K_2(t)$ lie in some order on the diagonal $X_2(t)Y_2(t)$. Note also that as t approaches t_0, $E_1(t)$ approaches the midpoint $M_1(t_0)$ of $X_1(t_0)Y_1(t_0)$, $E_2(t)$ approaches the midpoint $M_2(t_0)$ of $X_2(t_0)Y_2(t_0)$, and $K_1(t)$, $K_2(t)$ and $Bp(t)$ all approach $Bp(t_0)$.
Now the difference $Ar_1(t_0) - Ar_1(t)$ can be seen to be written as the alternating sum of the areas of four triangles:

$$\text{Area}(\Delta E_1(t)Z_2W_2) - \text{Area}(\Delta E_1(t)X_1(t_0)X_1(t)) + \text{Area}(\Delta E_2(t)X_2(t_0)X_2(t)) - \text{Area}(\Delta E_2(t)Z_1W_1),$$

where $Z_1W_1 \in \{Bp(t_0)K_1(t), Bp(t)K_2(t)\}$ and $Z_2W_2 \in \{Bp(t_0)K_2(t), Bp(t)K_1(t)\}$. It is possible that one or both of the triangles $\Delta E_1(t)Z_2W_2$ and $\Delta E_2(t)Z_1W_1$ might have area 0.
Let $F(t) = \dfrac{\text{Area}(\Delta E_1(t)Z_2W_2) - \text{Area}(\Delta E_1(t)X_1(t_0)X_1(t))}{(t - t_0)}$.
We can rewrite $F(t)$, using the fact that $\text{Area}(\Delta A_1B_1C_1) = .5\,|B_1 - A_1||C_1 - A_1|\sin(\angle B_1A_1C_1)$.
$F(t) = .5\,\dfrac{(|E_1(t) - Z_2||E_1(t) - W_2| - |E_1(t) - X_1(t_0)||E_1(t) - X_1(t)|)\,\sin((t - t_0)\theta_1)}{(t - t_0)}$.
As t approaches t_0, $E_1(t)$ approaches $M_1(t_0) = .5(X_(t_0) + Y_1(t_0))$, and Z_2 and W_2 both approach $Bp(t_0)$.
Now calculate the limiting value of $F(t)$ at $t = t_0$:
In line 3 below, we have assumed that $M_1(t_0)$ *is between* $Bp(t_0)$ *and* $Y_1(t_0)$. *If* $M_1(t_0)$ *is between* $Bp(t_0)$ *and* $X_1(t_0)$, *we would use* $|Bp(t_0) - M_1(t_0)| = |M_1(t_0) - X_1(t_0)| - |Bp(t_0) - X_1(t_0)|$. *Since the difference is squared, the result is*

the same.

$$\begin{aligned}
\lim_{t\to t_0} F(t) &= .5(|M_1(t_0) - Bp(t_0)|^2 - |M_1(t_0) - X_1(t_0)|^2) \lim_{t\to t_0} \frac{\sin((t-t_0)\theta_1)}{(t-t_0)} \\
&= .5\,(|M_1(t_0) - Bp(t_0)|^2 - |M_1(t_0) - X_1(t_0)|^2)\,\theta_1 \\
&= .5\,\theta_1\,((|Bp(t_0) - X_1(t_0)| - |M_1(t_0) - X_1(t_0)|)^2 - |M_1(t_0) - X_1(t_0)|^2) \\
&= .5\,\theta_1\,((-2\,|Bp(t_0) - X_1(t_0)||M_1(t_0) - X_1(t_0)|) + |Bp(t_0) - X_1(t_0)|^2) \\
&= .5\,\theta_1\,(|Bp(t_0) - X_1(t_0)|(|Bp(t_0) - X_1(t_0)| - 2|M_1(t_0) - X_1(t_0)|)) \\
&= .5\,\theta_1\,(|Bp(t_0) - X_1(t_0)|(|Bp(t_0) - X_1(t_0)| - |Y_1(t_0) - X_1(t_0)|)) \\
&= -.5\,\theta_1\,|Bp(t_0) - X_1(t_0)||Bp(t_0) - Y_1(t_0)|
\end{aligned}$$

For the other half of $\frac{Ar_1(t_0) - Ar_1(t)}{(t-t_0)}$, $G(t) = \frac{\text{Area}(\Delta E_2(t)Z_1W_1) - \text{Area}(\Delta E_2(t)X_2(t_0)X_2(t))}{(t-t_0)}$, a similar calculation can be made to calculate that the limit of $G(t)$ at t_0 is $.5\,\theta_1\,|Bp(t_0) - X_2(t_0)||Bp(t_0) - Y_2(t_0)|$.
Hence $Ar_1'(t_0) = .5\,\theta_1\,(|Bp(t_0) - X_2(t_0)||Bp(t_0) - Y_2(t_0)| - |Bp(t_0) - X_1(t_0)||Bp(t_0) - Y_1(t_0)|)$ for any $t_0 \in [0,1]$. The second form of the derivative follows from the chain rule. □

This result has useful corollaries. We had already observed in section 1 that the area map $AR(\phi)$, $\phi \in [0, \pi/2]$ defined on a convex polygon is continuous. It is in fact differentiable (with a piecewise continuous second derivative), as the first corollary states.

Corollary 1. *The area map AR on a convex polynomial $\mathcal{P}$ is differentiable when considered as a function of ϕ, and has a continuous second derivative except at the vertices of $\mathcal{P}$.*

Proof. As noted above, the area function AR on $\mathcal{P}$ is constructed piecewise from the area functions Ar_1 on Q_i and $Q_i^\perp$ by adding a constant to each piece. Hence the derivative of AR is well defined except possibly at the endpoints Q_i. However by theorem 8, the derivatives of each piece (when taken with respect to ϕ) agree at the common endpoints and so AR' is well defined everywhere. Also, since each piece of AR' is defined by an algebraic function of the sine and cosine of ϕ, AR'' is well defined and continously differentiable except possibly at the endpoints Q_i, where it may not be well-defined. □

To illustrate, we examine the quadrisections of the quadrangle $\mathcal{P}$ shown in Figure 7.

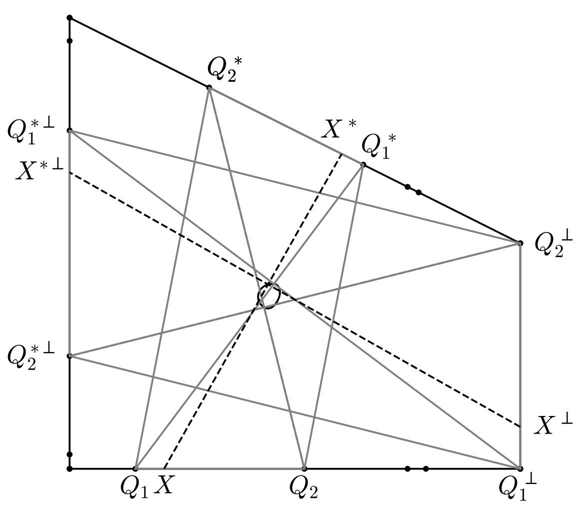


Figure 7.

$\mathcal{P}$ has only one quadrisection showing as the dashed cross in the second covering pair of orthogonal trapezoids of $\mathcal{P}$. The graphs of the area function and its derivative in Figure 8 confirm this.

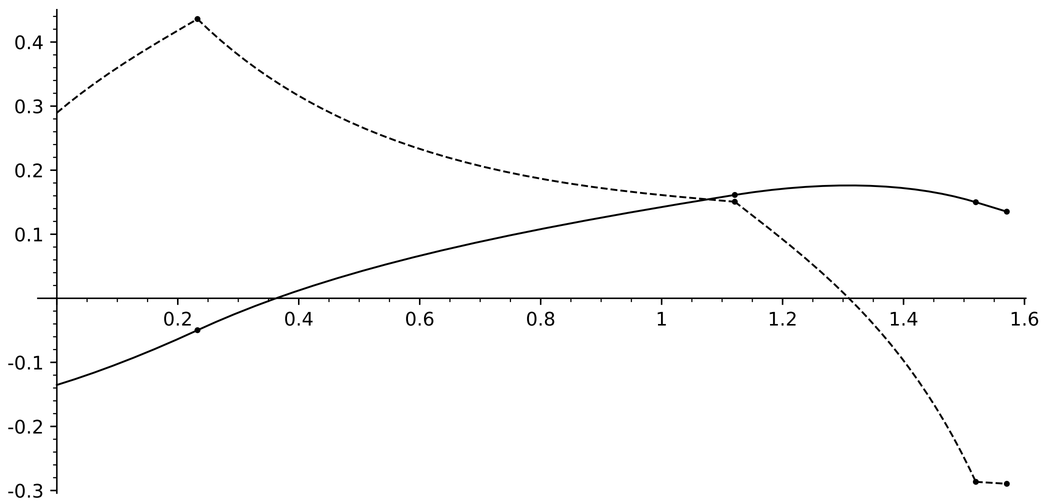


Figure 8.

For the next corollary, we use the notations $f(\alpha_i, \beta_i, t) = \dfrac{\sin(\alpha_i)\sin(\beta_i)}{\sin(\alpha_i + t\theta_1)\sin(\beta_i + t\theta_1)}$ and $a_i = .5|A_i - C_i|$for $i \in 1, 2$.

Corollary 2. $Ar_1'(t) = (a_2 f(\alpha_2, \beta_2, t))^2 - (a_1 f(\alpha_1, \beta_1, t)))^2 - (|M_2(t) - Bp(t)|^2 - |M_1(t) - Bp(t)|^2)$.

Proof. Note that $|Y_i(t) - X_i(t)| = |X_i(t) - Bp(t)| + |Y_i(t) - Bp(t)|$ for all t and $i \in 1, 2$. So $|X_i(t) - Bp(t)|$ and $|Y_i(t) - Bp(t)|$ are $.5|X_i(t) - Y_i(t)| - |Bp(t) - M_1(t)|$ and $.5|X_1(t) - Y_1(t)| + |Bp(t) - M_1(t)|$ in some order. Hence

$$\begin{aligned} |X_i(t) - Bp(t)||Y_i(t) - Bp(t)| &= (.5\theta_1|X_i(t) - Y_i(t)| - |Bp(t) - M_i(t)|)(.5\theta_1|X_i(t) - Y_i(t)| + |Bp(t) - M_i(t)|) \\ &= .25|X_i(t) - Y_i(t)|^2 - |Bp(t) - M_i(t)|^2 \end{aligned}$$

But $.25|X_i(t) - Y_i(t)|^2 = a_i^2 f(\alpha_i, \beta_i, t)$ for $i \in 1, 2$ by Theorem 3, and the result follows. □

What Corollary 2 shows is that the derivative of the area function is a quarter of the difference of the squares of the lengths of the diameters reduced by the difference of the squares of the deviations of the Bernoulli point $Bp(t)$ from the respective midpoints of the diameters.
We want to investigate the case when the Bernoulli point doesn't deviate from the midpoints at all. Suppose the orthogonal trapezoids $\mathcal{T}_1$ and $\mathcal{T}_2$ we started this section with satisfy this condition. Then $M_1(t) = Bp(t) = M_2(t)$ for all $t \in [0, 1]$. Then by Theorem 2, $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, that is, $\mathcal{T}_1$ and $\mathcal{T}_2$ are in fact orthogonal parallelograms whose first diagonals intersect at their respective midpoints.

Theorem 9. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are orthogonal parallelograms whose first diagonals intersect at their respective midpoints, then*
1) if $a_2 \cot(\alpha_1) = a_1 \cot(\alpha_2)$, then Ar_1 is increasing, (constant, decreasing) if $a_1 < a_2$ ($a_1 = a_2$, $a_1 > a_2$).
2) Otherwise, $Ar_1'(t) = 0$ exactly once in $(0, 1)$ when and only when $Ar_1'(0)Ar_1'(1) < 0$, Ar_1 is increasing (decreasing) on $[0, 1]$ when $Ar_1'(0)$ and $Ar_1'(1)$ are non-negative (non-positive) and at least one is positive (negative).

Proof. By Theorems 8 and 3,

$$(**)\ Ar_1'(t) = (a_2 \frac{\sin(\alpha_2)}{\sin(\alpha_2 + t\theta_1)})^2 - (a_1 \frac{\sin(\alpha_1)}{\sin(\alpha_1 + t\theta_1)})^2$$

. Factor this and note that the sign of Ar_1' is the sign of $(a_2 \frac{\sin(\alpha_2)}{\sin(\alpha_2 + t\theta_1)}) - a_1(\frac{\sin(\alpha_1)}{\sin(\alpha_1 + t\theta_1)})$. So write this as a simple fraction, note the denominator is positive, and so the numerator $a_2 \sin(\alpha_2)\sin(\alpha_1 + t\theta_1) - a_1 \sin(\alpha_1)\sin(\alpha_2 + t\theta_1)$ carries the sign of Ar_1'. Simplify this using the addition formula for sin, obtaining the expression

$$a_2 \sin(\alpha_2)\sin(\alpha_1)\cos(t\theta) + a_2 \sin(\alpha_2)\cos\alpha_1 \sin(t\theta) - a_1 \sin(\alpha_1)\sin(\alpha_2)\cos(t\theta) - a_1 \sin(\alpha_1)\cos(\alpha_2)\sin(t\theta)$$

Divide by the positive number $\cos(t\theta)\sin(\alpha_1)\sin(\alpha_2)$ and collect like terms to get

$$(*)\ \ (a_2 - a_1) + (a_2\cot(\alpha_1) - a_1\cot(\alpha_2))\tan(t\theta)$$

.

In case 1), if $a_1\alpha_2 - a_2\alpha_1 = 0$, then the sign of Ar_1' is $a_2 - a_1$, so is constant for all $t \in [0,1]$. Hence Ar_1 is increasing, constant, or decreasing according to the sign of $a_2 - a_1$.
In case 2), (*) has a unique solution $t_1 = \frac{1}{\theta_1}\arctan(\frac{a_1 - a_2}{a_2\cot(\alpha_1) - a_1\cot(\alpha_2)})$ if $Ar_1'(0)Ar_1'(1) < 0$, then either Ar_1 changes from increasing to decreasing or from decreasing to increasing on $[0,1]$. Hence the solution to (*) t_1 must lie in $(0,1)$. If $Ar_1'(0) = 0$ or $Ar_1'(1) = 0$, so Ar_1 is increasing (decreasing) on $[0,1]$ if $Ar_1'(1) > 0$ $Ar_1'(1) < 0$) □

We have the tools to calculate the quadrisections of any convex polygon with **central symmetry**, that is, reflection thru some point carries $\mathcal{P}$ onto $\mathcal{P}$.

4.1 Quadrisections of convex polygons with central symmetry.

Throughout this section, $\mathcal{P}$ is a convex polygon with central symmetry. Thus $\mathcal{P}$ might be a regular polygon with an even number of sides or various linear distortions of one. For example, it might be a parallelogram.
Here is a useful fact.

Theorem 10. *Each pair of the covering trapezoids of $\mathcal{P}$ consists of parallelograms.*

Proof. First note that the quadrangles with central symmetry are the parallelograms. But the covering trapezoids of $\mathcal{P}$ have central symmetry. □

$\mathcal{P}$ by necessity must have an even number of vertices, say $2n$. By relabeling, scaling and rotation and translation, we can situate $\mathcal{P}$ so that $P_0 = (1,0)$, $P_n = (-1,0)$ and $\mathcal{P}$ lies in the unit disk. We can see that each line through the origin contains a bisector of $\mathcal{P}$. So there are n bisectors P_iP_{n+i} for $i = 0,\cdots,n-1$, together with up to n more bisectors $P_i^\perp P_{n+i}^\perp$ $i = 0,\cdots,n-1$. Some of the perpendicular bisectors will already be counted, if their endpoints are vertices. So $\mathcal{P}$ is covered by $m <= n$ pairs of orthogonal parallelograms, $Q_i, Q_i^\perp, i = 1,\cdots,m$.
We can make a general statement about the number of quadrisections of $\mathcal{P}$.

Theorem 11. *If $\mathcal{P}$ has central symmetry with $2n$ vertices and $m \le n$ covering pairs of parallellograms, then it has at most $m+1$ quadrisections where we in the case that in some covering pair with a quadrisection, $Ar_1'(t) = 0$ for all $t \in [0,1]$ we count all the quadrisections as one.*

Proof. Note that for each of the covering pairs, Theorem 9 tells us that that $AR'(t) = 0$ at most m times where we count the case where $Ar_1'(t) = 0$ over the whole interval as one. Consequently, since AR is differentialble, $AR(t)$ will take on a given value C at most $m+1$ times counting intervals where $AR(t) = C$ as one. So $\mathcal{P}$ has at most $m+1$ quadrisections, counting intervals of quadrisections as one. □

We can completely describe the quadrisections of the regular polygons with an even number of sides. Let $\mathcal{R}_{2n}$ $(n > 1)$be the regular polygon with $2n$ sides. We position $\mathcal{R}_{2n}$ in the unit disk so that its vertices are the $2n$ roots of unity, $P_i = (\cos(\frac{i}{n}\pi), \sin(\frac{i}{n}\pi))$, $i = 0,\cdots,2n-1$.
If $n = 2m+1$ $(m \ge 1)$,

Theorem 12. *Let $\mathcal{R}_{2n}$ be the regular polygon with $2n$ sides. If n is even, then every cross in $\mathcal{R}_{2n}$ is a quadrisection. If n is odd, then $\mathcal{R}_{2n}$ has exactly n quadrisections, namely the crosses which contain a pair of vertices of $\mathcal{R}_{2n}$.*

Proof. If $n = 2m$ ($m \geq 1$), then $\mathcal{R}_{2n}$ is covered by m pairs of orthogonal parallelograms $\mathcal{Q}_i = P_i P_{i+1} P_{i+n} P_{i+n+1}$, $\mathcal{Q}_i^\perp = P_{i+m} P_{i+m+1} P_{i+m+n} P_{i+m+n+1}$ for $i = 0, \cdots, m-1$. These pairs are all congruent with each other by a rotation about the origin thru multiples of the the angle $\frac{\pi}{n}$. In each pair, we have $a_1 = a_2 = 1$ and $\alpha_1 = \alpha_2 = \frac{\pi}{2n}$. Hence by Theorem 9, AR is constant and every cross is a quadrisection.
If $n = 2m+1$ ($m \geq 1$), then $\mathcal{R}_{2n}$ is covered by n pairs of orthogonal parallelograms $\mathcal{P}_i = P_i M_i P_{i+n} M_{i+n+1}$, $\mathcal{P}_i^\perp = M_{i+m} P_{i+m+1} M_{i+m+n} P_{i+m+n+1}$ for $i = 0, \cdots, n-1$, where $M_i = .5(P_i + P_{i+1})$. These pairs are all congruent with each other by a rotation about the origin through the angle π/n, and a symmetry argument shows that the cross $P_0 P_n$, $M_m M_{m+n}$ is a quadrisection. We compute that $a_1 = 1$, $a_2 = \sin(\frac{n-1}{n}\frac{\pi}{2}) < a_1$, and $\alpha_1 = \frac{\pi}{2}\frac{n-2}{n}$, $\alpha_2 = \frac{\pi}{2}$. Plugging these values into (**) in the proof of Theorem 9, we get that $Ar_1'(0) = \cos(\frac{\pi}{2n}) - 1 < 0$ and $Ar_1'(1) = \frac{\cos(\pi/(2n)) - \cos(\pi/n)}{\cos(\pi/(2n))} > 0$. So by Theorem 9, $Ar_1' = 0$ once between 0 and 1. (Actually at $t = .5$). □

Remark. It is interesting that if $\mathcal{R}_{2n}$ is distorted by even the slightest by a linear map, then the resulting polygon has only one quadrisection.

4.2 Some results on quadrisections of quadrilaterals.

We can describe completely the quadrisections of a parallelogram.

Theorem 13. *Let $\mathcal{P}$ be a parallelogram. Then if $\mathcal{P}$ is a square, every cross yields a quadrisection. Otherwise, $\mathcal{P}$ has exacly one quadrisection.*

Proof. $\mathcal{P}$ is situated in the unit disc with $P_0 = (1, 0)$, $P_2 = (-1, 0)$. Let $P_1 = (r\cos(\theta), r\sin(\theta))$. By the law of sines, $r = \frac{\sin(\alpha_1)}{\sin(\alpha_1 + \theta_1)}$. Note by assumption, $r <= 1$ and we can assume $\theta <= \pi/2$, so that P_1 is in the first quadrant.
Calculate that $P_0^\perp = (0, \frac{r\sin(\theta)}{r\cos(\theta)+1})$ and $P_2^\perp = (0, -\frac{r\sin(\theta)}{r\cos(\theta)+1}) = P_0^{*\perp}$.
Case 1) $\theta = \pi/2$: then $P_2 = P_0^\perp = (r, 0)$ and $P_3 = P_2^\perp = (-r, 0)$, so $\mathcal{P}$ is a rhombus, and is covered by one pair of orthogonal parallelograms, $\mathcal{Q}_0 = P_0 P_1 P_2 P_3$ and $\mathcal{Q}_0^\perp = u P_1 P_2 P_3 P_0$. So $a_1 = 1$, $a_2 = r$If $r = 1$ then $\alpha_1 = \alpha_2 = \pi/4$, (so $\mathcal{P}$ is a square)and Ar_1 is constant on $[0, 1]$ by Corollary 9. So every cross in a square determines a quadrisection. (Note. This can be verified directly by observing that every cross in a square divides it into 4 congruent quadrangles.).
If $r < 1$, then the cross $P_0 P_2$, $P_1 P_3$ is a quadrisection. But $\alpha_1 < \pi/4$, so $\alpha_2 = \pi/2 - \alpha_1 > \alpha_1$. Hence by Corollary 9, $Ar_1(t)$ decreases from $Ar_1(0)$ to $Ar_1(.5)$, then increases back to $Ar_1(1) = Ar_1(0) = .25\text{Area}\mathcal{P}$.
Case 2) $\theta < pi/2$: then $P_0^\perp \neq P_1$ and $P_2^\perp \neq P_3$, so $\mathcal{P}$ is covered by 2 pairs of orthogonal parallelograms. In the first pair, $\mathcal{Q}_0$, $\mathcal{Q}_0^\perp$, we have that $a_1 = 1 > a_2 = \frac{r\sin(\theta)}{1+\cos(\theta)}$ and $a_1\cot(\alpha_2) = \cot(\alpha_2) = \frac{a_2}{a_1} = a_2$ and $a_2\cot(\alpha_1) = a_2\frac{r\sin(\theta)}{1 - r\cos(\theta)} = a_2\frac{r^2\sin(\theta)^2}{1 - r^2\cos(\theta)^2} = a_2$. So by Corollary 9, $AR_1(t)$ is decreasing on $[0, 1]$. Since $AR_1(0) > .25\text{Area}\mathcal{P} > AR_1(1)$, there is exactly one quadrisection of $\mathcal{P}$ occurring in the first covering pair. In the second covering pair, one works out that $a_1 > a_2$, α_2 $Ar_1(0) = Ar_1(1)$ □

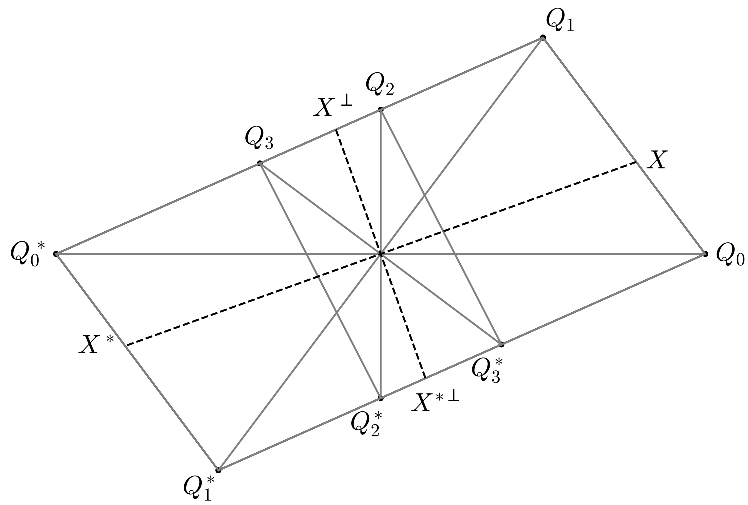


Figure 9.

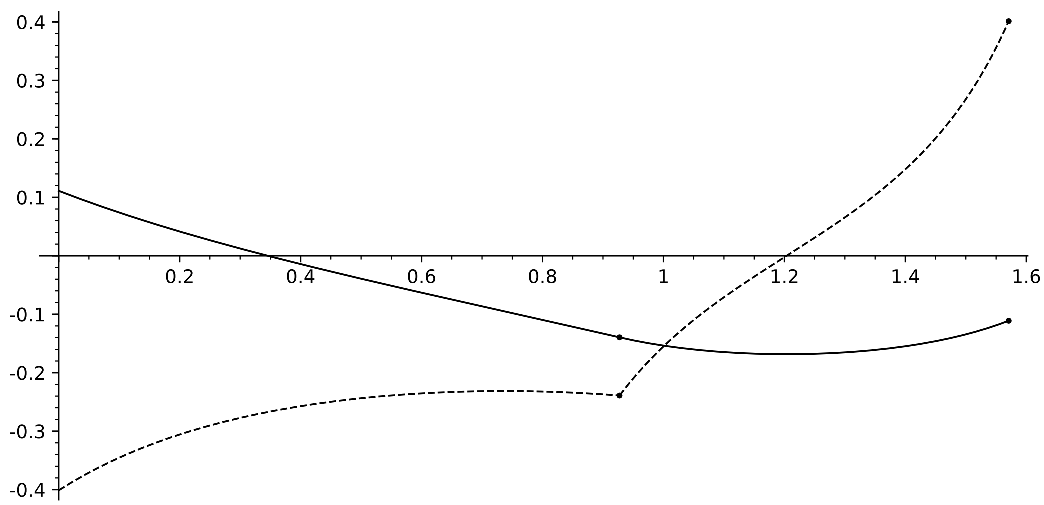


Figure 10.

4.3 The quadrisections of regular polygons with an odd number of sides.

Let $\mathcal{L}$ be the line through The orthogonal pair $\mathcal{T}_1$ and $\mathcal{T}_2$ is called a **reflective orthogonal pair** if 1)

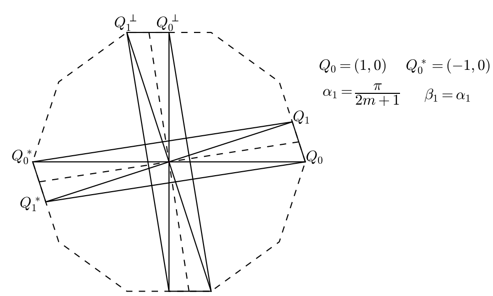
Corollary 3. *Suppose that in both $\mathcal{T}_1$ and $\mathcal{T}_2$ we know that there is a constant $0 \le c < 1$ such that $|Bp(t) - Md_1(t)| < c\dfrac{|X_1(t) - Y_1(t)|}{2}$ and $|Bp(t) - Md_2(t)| < c\dfrac{|X_2(t) - Y_2(t)|}{2}$ for all $t \in [0, 1]$. Then the sign of $Ar_1(t)$ is the sign of $(a_2 \cot(\alpha_1) - a_1 \cot(\alpha_2)) \tan(t\theta_1)$ and the conclusion of Theorem 9 holds.*

Proof. By □

5 Quadrisection of the the regular polygons

Fix n and realize the regular n gon as the polygon $\mathcal{R}_n$ with vertices the vectors $R_i = (\cos(i\theta_n), \sin(i\theta_n))$, where $\theta_n = \dfrac{2\pi}{n}$, and $i = 0, \cdots, n-1$.

There is a fundamental pair of orthogonal trapezoids $\mathcal{T}_n$ and $\mathcal{T}_n^{\perp}$ for $\mathcal{R}_n$ which when rotated k_n times forms the covering of $\mathcal{R}_n$ by orthogonal trapezoids, where k_n is n if n is odd, and k_n is m is $\frac{n}{4}$ if n is divisible by 4, or k_n is $\frac{n}{2}$ if n is an odd multiple of 2.
The description of the quadrisections of $\mathcal{R}_n$ divides naturally into four cases, the **two even cases**: $n = 4m$ or $n = 4m + 2$, $m = 1, 2, \cdots$ and the **two odd cases**: $n = 4m - 1$ or $n = 4m + 1$, $m = 1, 2, \cdots$.
The even case $n = 4m$ is easy. These polygons $\mathcal{R}_{4m}$ all have *central symmetry*, that is, reflection through the origin carries $\mathcal{P}_{4m}$ onto itself. Hence the bisectors of $\mathcal{P}_{4m}$ are the lines through $(0,0)$. Also, rotation of $\mathcal{P}_{4m}$ by 90^o about $(0,0)$ carries $\mathcal{P}_{4m}$ onto itself, and so each pair of perpendicular bisectors of $\mathcal{P}_{4m}$ forms a quadrisection. The Bernoulli necklace is degenerate, and has one bead.
The other even case $n = 4m + 2$ is slightly harder. We no longer have central symmetry for $\mathcal{P}_{4m+2}$. However, $\mathcal{R}_{4m+2}$ has $4m + 2$ lines of symmetry, namely the $2m + 1$ lines through the vertices P_i and P_{2m+1+i}, $i = 0, \cdots, 2m$, together with the $2m + 1$ lines through the midpoints Md_i, Md_{2i+1} of the sides P_iP_{i+1} and $P_{2m+1+i}P_{2m+2+i}$. Thus we can work out that there are $k_n = 2m + 1$ pairs of orthogonal trapezoids $\mathcal{Q}_i, \mathcal{Q}_i^{\perp}$ covering $\mathcal{R}_{4m+2}$. The first pair consists of the trapezoids $\mathcal{Q}_0 = P_0M_0P_{2m+1}M_{2m+1}$ and $\mathcal{Q}_0^{\perp} = M_mP_{m+1}M_{3m+1}P_{3m+2}$. The remaining $2m$ pairs can be obtained from this pair by rotating the first pair about the origin through the angles, $2i\pi/n$ for $i = 1, \cdots, 2m$.
What makes this case slightly harder is that now we have to show that there are no other quadrisections. To prove this we will use Theorem 8. Since all the $k_{4m} = 2m + 1$ pairs of orthogonal trapezoids $\mathcal{Q}_i, \mathcal{Q}_i^{\perp}$ are congruent, we need only consider the first pair $\mathcal{Q}_0, \mathcal{Q}_0^{\perp}$. We can calculate the five determining parameters for $\mathcal{Q}_0$: $A_1 = (1, 0)$, $C_1 = (-1, 0)$, $\alpha_1 = \frac{2m}{2m+1}\frac{\pi}{2}$, $\beta_1 = \alpha_1$, and $\theta_1 = \frac{1}{2m+1}\frac{\pi}{2}$. $\mathcal{Q}_0^{\perp}$ is the reflection of $\mathcal{Q}_0$ about the line through $(0,0)$ and $(\cos(\pi/4 + \theta_1/2), \sin(\pi/4 + \theta_1/2))$.
We will show that for the first pair, the graph of the Area function Ar_1 (see Theorem 7) is decreasing, then increasing. This shows that there are no quadrisections between
The regular n gon has n lines of symmetry $_i$, $i = 0, \cdots, n-1$, where $_i$ is the line through $(0,0)$ and P_i. Each line of symmetry yields a quadrisection (the bisector of $\mathcal{P}_n$ perpendicular to $_i$ together with $_i$ is a quadrisection of $\mathcal{P}_n$. But these n quadrisections are all congruent under rotation about $(0,0)$, so we only need to compute one of them, say the one having $_0$ as one of the bisectors. Because of the symmetry about the x-axis, we need only find the vertical diameter XY which bisects the upper half UH of the polygon. The vertices of UH consist of



$\mathcal{Q}_0$ and $\mathcal{Q}_0^{\perp}$ for $\mathcal{P}_6$

$P_0, \cdots, P_n, M$ where $M = 0.5\,(P_n + P_{n+1})$. Let $m = \text{floor}(\frac{n}{2})$. Then we see that $Y = P_m + t(P_{m+1} - P_m)$ for some $t \in (0, 1)$. The area A of UH to the right of XY is the sum of the areas of trapezoids and can be written as a quadratic function of t: $A = B(t) + C$ where $A = \frac{1}{8}(n + \frac{1}{2})\sin(\theta_n)$, $B(t) = (\Re(P_m) - \Re(Y))\frac{(\Im(Y) + \Im(P_m))}{2}$ and $C = \sum_{j=0}^{m-1}(\Re(P_j) - \Re(P_{j+1}))\frac{(\Im(P_j) + \Im(P_{j+1}))}{2}$.
So the quadrisection is obtained by solving the quadratic equation: $B(t) = A - C$. Letting $cm = \cos(m\theta_n)$, $cm1 = \cos((m+1)\theta_n)$, $sm = \sin(m\theta_n)$ and $sm1 = \sin((m+1)\theta_n)$, we can write $\Re(P_m) = cm$, $\Im(P_m) = sm$, $\Re(P_{m+1}) = cm1$ and $\Im(P_{m+1}) = sm1$, and the quadratic equation can be written
$t^2 + \frac{2sm}{sm1 - sm}t + \frac{2(A - C)}{(cm1 - cm)(sm1 - sm)} = 0$. The solution is
$t = \frac{sm}{sm - sm1} \pm \sqrt{(\frac{sm}{sm - sm1})^2 + \frac{2(A - C)}{(cm - cm1)(sm - sm1)}}$ depending on whether n is odd or even.
When this value for t is used to calculate Y, we see that $x_n = \Re(Y) > 0$ when n is even, and $x_n = \Re(Y) < 0$ when n

is odd. We can also see that the sequence x_n starts small at $x_2 = 0.0147$ and appears to converge linearly to 0: $x_3 = -0.005505$, $x_4 = 0.002617$, $x_5 = -0.001441$, $x_6 = 0.0008755$, $x_7 = -0.000571$ See the Sagemath calculations and diagrams below.

6 Elementary facts from linear algebra and analytic geometry.

EF 1. The area of a triangle with vertices A, B, C labeled counterclockwise is one-half the determinant of the matrix with first row $B - A$ and second row $C - A$, written $\frac{1}{2}\begin{vmatrix} B - A \\ C - A \end{vmatrix}$. Also the area can be written as

$\frac{1}{2}|B - A||C - A|\sin(\theta)$ where $\theta = \angle BAC$ and $|X - Y|$ is the (Euclidean) norm of the vector $X - Y$.
Note that this can be rewritten several ways using basic properties of the determinant function. For example, $\frac{1}{2}\begin{vmatrix} B - A \\ C - A \end{vmatrix} = \begin{vmatrix} A - C \\ \frac{1}{2}(B - A) \end{vmatrix}$. Also, note that if one or more of the vertices is moved continuously, the area of the triangle changes continuously since the determinant function is continuous.
EF 2. Two lines AD and BC are parallel if and only if $\begin{vmatrix} D - A \\ B - C \end{vmatrix}$ is 0. This follows from EF 1, since AD and BC are parallel if and only if $B - C = m(A - D)$ for some m, and $\begin{vmatrix} D - A \\ m(D - A) \end{vmatrix}$ is 0.
EF 3. The angle θ between two vectors U and V is acute (right, obtuse) if and only $U \cdot V$ is positive (0, negative). This follows from the fact that $U \cdot V = |U|\,|V|\,\cos\theta$.
For any angle α, define $U_\alpha = (\cos\alpha, \sin\alpha)$, the unit vector pointing in the direction α. Then, by the trig addition formulas,
EF 4. $U_\alpha \cdot U_\beta = \cos(\alpha - \beta)$ and $U_\alpha \cdot U_{\beta+\pi/2} = \sin(\alpha - \beta) = \begin{vmatrix} U_\beta \\ U_\alpha \end{vmatrix}$.
EF 5. Given an angle ϕ and a point P_0, define $M = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$. Then **translation by** P_0, the map $tr_{P_0} : P \to P + P_0$, and **rotation by** ϕ, the map $rot_\phi : P \to M\,P$ are isometries of the plane.
If we parameterize all the orthogonal pairs $\mathcal{Q}_i, \mathcal{Q}_i^\perp$, $i = 0, \cdots, m-1$ of the polygon $\mathcal{P}$ with the angle parameterization, we can define

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