A model for the space of convex quadrilaterals

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Abstract. This paper describes a 'model' for the space of convex quadrilaterals, that is, a set $\mathbb{Q}$ of convex quadrilaterals in the plane which gives a 'cross-section' of the equivalence classes of similar convex quadrilaterals, in the sense that every convex quadrilateral is similar to exactly one member of $\mathbb{Q}$. It is also has the property that quadrilaterals which have nearly the same vertices are close to each other in the model in the Hausdorff metric.

A parameterization of the model with the 4-cell is defined and used to describe and investigate various classes of quadrilaterals. For example, the trapezoids form a 3-dimensional closed subset of the model which separates the model into 3 disjoint open sets, each homeomorphic with a 4-cell. **Sage-Math on CoCalc.com** is used to generate the figures and latex for this paper, and also to make the **Sage-Cell Interacts** to explore the model, available at *https://sagelets.cocalc.com/quad_index.html*.

Quadrisections of convex polygons are discussed and questions are posed about the maximum finite number of quadrisections of a convex quadrilateral.

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1. Introduction

Recall that a *convex quadrilateral* is a polygon with 4 vertices such that the intersection of the diagonals lies in the interior of each . There are many different kinds of convex quadrilaterals: *trapezoid*, *parallelogram*, *rhombus*, *rectangle*, *kite*, *cyclic quadrilateral* and many others. Over the centuries, there have been numerous classification schemes proposed. Martin Josefsson [1] has given a good summary of these in his paper *On the classification of convex quadrilaterals* and has proposed another interesting classification. Also, the paper by Ahtziri Gonzalez and Jorge L. Lopez-Lopez [2] has established some of our conclusions using more sophisticated mathematics.

This paper describes a 'model' for the space of convex quadrilaterals, that is, a set $\mathbb{Q}$ of convex quadrilaterals in the plane which gives a 'cross section' of the equivalence classes of similar convex quadrilaterals in the sense that every convex quadrilateral is similar to exactly one member of $\mathbb{Q}$. This model also has the property that quadrilaterals which have nearly the same vertices are close to each other. However, it unavoidably also has nearly congruent quadrilaterals which are far apart.

We modeled the space of triangles in [3]. By scaling and a Euclidean motion, we can place each triangle so that the its vertices are $A = (1, 0)$, $B = (x, y)$, and $C = (-1, 0)$, where $x \geq 0$, $y > 0$, and $(x+1)^2 + y^2 \leq 4$, that is, B is in first quadrant above the x-axis and $|BC| \leq 2$, where $|BC|$ is the Euclidean distance between points B and C. That model is 2 dimensional and is a disk with a closed arc removed from the boundary (see Figure 1). We also investigated how the various types of triangles were situated in the model.

For example, the boundary of the model consists of an open interval of isosceles triangles with the equilateral triangle at the midpoint with the tall isosceles triangles to the right and the short isosceles triangles to the left. Also, the right triangles forms an arc (half-open) ending at the isosceles right triangle and separating the model into two components, one being all triangles with an obtuse angle and the other being all triangles with no obtuse angle.

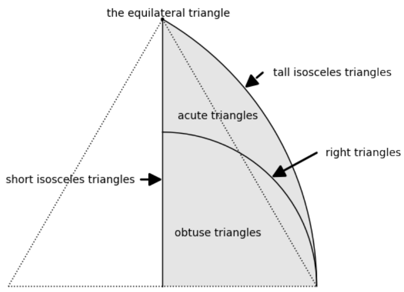

Figure 1: Space of triangles.

We were primarily interested in identifying the triangles with one, two or three *quadrisections* (perpendicular segments dividing a triangle into four equal areas), and found that almost all triangles have only one quadrisection save a small open (in the model) set of triangles with three quadrisections about the equilateral triangle. The boundary of this is a closed arc separating the triangle with three quadrisections from those with only one quadrisection, starting at the only isosceles triangle with two quadrisections and consisting of scalene triangles with two quadrisections except for the other endpoint which is an isosceles triangle with one quadrisection.

Any model for the convex quadrilaterals is 4 dimensional: Just fix two vertices and let the other two roam subject to the conditions that the quadrilateral must be convex and not degenerate to a triangle or segment. Also we don't want to have two congruent quadrilaterals in our model, but every quadrilateral is to be similar to exactly one quadrilateral in our model.

2. Description of the Model

We will build our model for the convex quadrilaterals from the set $\mathbb{Q}$ of quadrilaterals $ABCD$ which satisfy the following conditions:

Conditions for membership in $\mathbb{Q}$: *$A = (1, 0)$, $C = (-1, 0)$, $B = (x, y)$ with $x \geq 0, y > 0$, and $D = (z, w)$ with $w < 0$, $z \leq x$,and $(x-z)^2+(y-w)^2 \leq 4$. Also, we require that the intersection $X = (u, 0)$ of AC and BD satisfy $0 \leq u < 1$.*

In other words, for each member $ABCD$ of $\mathbb{Q}$, the diagonal BD has length no more than 2, AX has length no more than 1, and $\angle AXB$ is no more than 90 degrees.

If $ABCD$ is in $\mathbb{Q}$, we call AC and BD the **major** and **minor** diagonals of $ABCD$.

We claim that each convex quadrilateral Q is similar to one or possibly two members of $\mathbb{Q}$. By scaling, rotating, reflecting and translating we can move the quadrilateral so that one diagonal is AC with $A = (1, 0)$ and $C = (-1, 0)$ in eight, four, two, or one orientations, depending on its symmetries.

One of these orientations, or rarely two of them, satisfy the conditions for membership in $\mathbb{Q}$ above.

In the rare cases where there are duplicates copies of Q in $\mathbb{Q}$, we establish rules for which one to remove from $\mathbb{Q}$ below.

Here are typical examples of quadrilaterals with eight or four orientations.

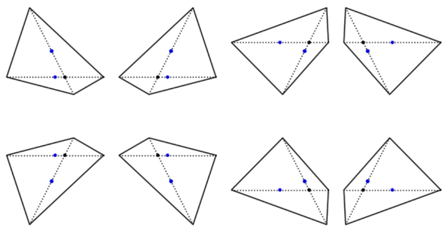

Figure 2: Eight orientations

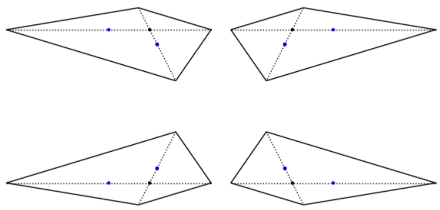

Figure 3: Four orientations

The duplicates occur when (1) the diagonals are perpendicular, (2) the minor diagonal bisects the major diagonal, or (3) the diagonals have the same length. Each case occurs on the **boundary** of $\mathbb{Q}$ and there is only one duplicate and there are good rules for elimination of one of the duplicates. A member $ABCD$ is in the **boundary** of $\mathbb{Q}$ if it is possible to move $ABCD$ out of $\mathbb{Q}$ by moving B or D by an arbitrarily small distance. The vast bulk of the quadrilaterals in $\mathbb{Q}$ lie in the **interior** of $\mathbb{Q}$ consisting of those $ABCD$'s in $\mathbb{Q}$ for which there is a positive number ϵ such that each quadrilateral $AB'CD'$ with $A = (1, 0)$, $C = (-1, 0)$, and B', D' within ϵ of B, D respectively lies in $\mathbb{Q}$.

The boundary of $\mathbb{Q}$ consists of three special classes of quadrilaterals. (1) **orthodiagonal** quadrilaterals, those where the diagonals are perpendicular, (2) **bimajor** quadrilaterals, those whose minor diagonal **bi**sects the **major** diagonal, and (3) **equidiagonal** quadrilaterals, those with diagonals of the same length. The first and last classes are named in Josefsson's[1] classification scheme. The term bimajor is of our making.

There is another class which has not been named before which is important in our discussion, namely, the class of quadrilaterals whose minor diagonal is bisected by its major diagonal. This class contains the parallelograms and the kites, but is much larger than either. We agree to call this class the **biminor** quadrilaterals.

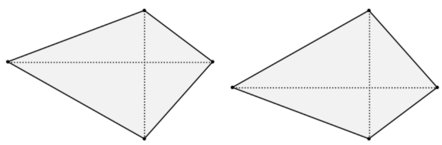

Figure 4: Duplicate orthodiagonal quadrilaterals in $\mathbb{Q}$

If $ABCD \in \mathbb{Q}$ is orthodiagonal, then it has a **twin** $AB'CD'$ obtained by reflecting $ABCD$ about the x-axis. These are congruent orthodiagonl quadrilaterals in $\mathbb{Q}$, and one must be deleted.

We could eliminate either of these from $\mathbb{Q}$, but must make a consistent choice. Our rule below will eliminate the one which sticks up further above the x-axis, that is, the one where $|BX| > |DX|$. Note that if $ABCD$ is it's own twin (ie $|BX| = |DX|$), then it is a **kite**.

For bimajor quadrilaterals there are pairs $ABCD$ and $AB'CD'$ which are congruent under a rotation of 180 degrees about the origin. Here again we will eliminate the quadrilateral where $|BX| > |DX|$. If $|BX| = |DX|$ (ie, $ABCD$ is biminor), then $ABCD$ is a **parallelogram**.

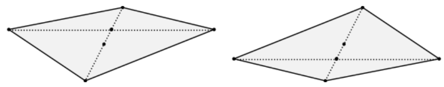

Figure 5: Duplicate bimajor quadrilaterals in $\mathbb{Q}$

Certain equidiagonal quadrilaterals in $\mathbb{Q}$ also have duplicates in $\mathbb{Q}$. There are of two types. (1) If $|AX| < |BX| < |DX|$. Then by reflecting $ABCD$ about the line which bisects $\angle BXA$ and following with a horizontal translation we can move BD onto AC yielding $AB'CD' \in \mathbb{Q}$ congruent to $ABCD$ with $|AX'| > |B'X'|$. We will eliminate from $\mathbb{Q}$ the one with $|AX| > |BX| >$. If $|BX| = |AX|$, then $|DX| = |CX|$ and by side angle side, $ABCD$ is an isosceles trapezoid, what we term later a **tall** isosceles trapezoid. (2) $|BX| > |DX|$. In this case, reflection through the bisector of $\angle BXA$ puts B' onto the wrong side of X, but rotation of 180 degrees about $(0, 0)$ followed by a horizontal translation carries $B'D'$ onto CA. Here we agree to eliminate the quadrilateral with $|DX| > |AX|$ Again, if $|AX|$ and $|DX|$ have the same length, there is only one member of $\mathbb{Q}$ and it is a **short isosceles trapezoid**.

Note: There is a sagelet in *https://sagelets.cocalc.com/QuadSpace.html* to check the duplications.

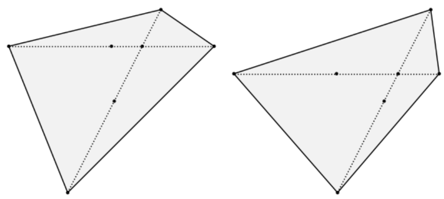

Figure 6: Equidiagonal duplicates (1)

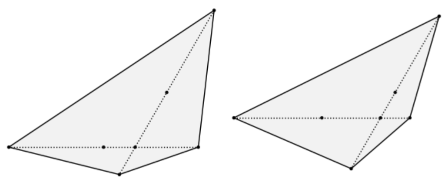

Figure 7: Equidiagonal duplicates (2)

To summarize, we have these

Elimination Rules:

(1) If $ABCD \in \mathbb{Q}$ is orthodiagonal and $|BX| > |DX|$, then eliminate $ABCD$ from $\mathbb{Q}$.

(2) If it is bimajor and $|BX| > |DX|$, then eliminate $ABCD$ from $\mathbb{Q}$.

(3) If it is equidiagonal with $|AX| < |BX| < |DX|$ or $|AX| < |DX| < |BX|$, then eliminate $ABCD$ from $\mathbb{Q}$.

At this point, the description of the model $\mathbb{Q}$ is complete. Now we will **parameterize** it and use the **parameter space** to help visualize the various classes of quadrilaterals and their relation to each other.

3. Parameterizing the Model.

From 2, recall that a typical element of $\mathbb{Q}$ has vertices $A = (1, 0)$, $B = (x, y)$, $C = (-1, 0)$, and $D = (z, w)$ where AC and BD intersect at $X = (u, 0)$ and x, y, z, w, u satisfy the restrictions $(x - z)^2 + (y - w)^2 \leq 4$, $0, z \leq x$, $w < 0 < y$, and $0 \leq u < 1$. In addition, by the elimination rules above, if $|BD| = 2$ then $|BX| \leq |AX|$, and if $\angle AXB$ is a right angle, then $|BX| \leq |DX|$.

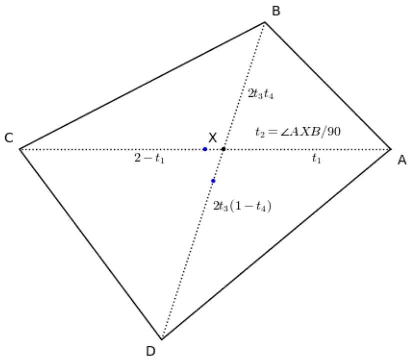

Figure 8: Parameters for $ABCD$.

We could use the coordinates of B and D, x, y, z, w to parameterize $ABCD$, but these are very inconvenient parameters to work with. We prefer working with parameters in the unit interval $[0, 1]$.

The most convenient set of parameters for a member $\mathcal{Q} = ABCD$ of $\mathbb{Q}$ we have found comes from the major and minor diagonals AC and BD of $\mathcal{Q}$ and their intersection point X. Associate with $\mathcal{Q}$ a unique 4-tuple (t_1, t_2, t_3, t_4) with $t_i \in (0, 1], i \in \{1, 2, 3, 4\}$ as follows:

$\mathbf{t_1} = \dfrac{\mathbf{2\,|AX|}}{\mathbf{|AC|}}$. So $t_1 \in (0, 1]$ when X is in the right hand half of AC.

$\mathbf{t_2} = \dfrac{\angle\mathbf{BXA}}{\mathbf{90}}$, $\angle BXA$ is measured in degrees. So $t_2 \in (0, 1]$ determines $\angle AXB$.

$\mathbf{t_3} = \mathbf{\frac{1}{2}|BD|}$. So since $2\,t_3 = |BD| \leq |AC| = 2$, $t_3 \in (0, 1]$.

$\mathbf{t_4} = \dfrac{\mathbf{|BX|}}{\mathbf{|BD|}}$, so $t_4 \in (0, 1)$. Note that $|DX| = 2\,t_3\,(1 - t_4)$.

Note that $t_1 = 1$ if and only if $\mathcal{Q}$ is bimajor, $t_2 = 1$ if and only if $\mathcal{Q}$ is orthodiagonal, $t_3 = 1$ if and only if $\mathcal{Q}$ is equidiagonal and $t_4 = .5$ if and only if $\mathcal{Q}$ is biminor.

This sets up a 1-1 correspondence $f : \mathcal{Q} = ABCD \to (t_1, t_2, t_3, t_4)$ from $\mathbb{Q}$ to the **parameter space** $\mathbb{P}$ of $\mathbb{Q}$, contained in the 4-cell $\mathbb{I}^4 = (0, 1] \times (0, 1] \times (0, 1] \times (0, 1)$. The correspondence is not onto $\mathbb{I}^4$, because of the elimination of certain duplicate equidiagonal, orthodiagonal, and bimajor quadrilaterals from $\mathbb{Q}$.

We eliminated the bimajor quadrilaterals from $\mathbb{Q}$ with $2\,t_3\,t_4 = |BX| > |AX| = 1$ or $t_4 \geq t_3\,t_4 > \dfrac{1}{2}$, so points $(1, t_2, t_3, t_4) \in \mathbb{I}^4$ with $\dfrac{1}{2} < t_4$ are not in $\mathbb{P}$. Also, we have eliminated the orthodiagonal quadrilaterals $ABCD$ with $2\,t_3\,t_4 = |BX| > |DX| = 2\,t_3\,(1 - t_4)$. So points $(t_1, 1, t_3, t_4) \in \mathbb{I}^4$ with $t_4 < \dfrac{1}{2}$ are not in $\mathbb{P}$. Finally, we eliminated the duplicate equidiagonal quadrilaterals ($t_3 = 1$) in $\mathbb{Q}$ with $t_1 = |AX| < |BX| = 2\,t_4 < |DX| = 2(1 - t_4)$ or

$|AX| < |DX| < |BX|$. So points $(t_1, t_2, 1, t_4) \in \mathbb{I}^4$ with $\dfrac{t_1}{2} < t_4 < 1 - \dfrac{t_1}{2}$ are not in $\mathbb{P}$.

So the parameter space $\mathbb{P}$ of $\mathbb{Q}$ is $\mathbb{I}^4 - (\mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3)$, where $\mathbb{B}_1$ is one-half of the 1-boundary in the t_1 coordinate, $\mathbb{B}_1 = \{1\} \times (0,1] \times (0,1] \times (0,.5)$, $\mathbb{B}_2$ is one half the 1-boundary in the t_2 coordinate, $\mathbb{B}_2 = (0,1] \times \{1\} \times (0,1] \times (0,.5)$, and $\mathbb{B}_3$ is a strange looking one half of the 1-boundary in the t_3 coordinate, $\mathbb{B}_3 = \bigcup_{t_1 \in (0,1]} \{t_1\} \times (0,1] \times \{1\} \times (0, t_1/2)$. See the diagram **Graph of the boundary of** $\mathbb{P}$.

So now the correspondence $f : \mathbb{Q} \to \mathbb{P}$ is 1-1 *and* onto. It is also true that f is a homeomorphism, that is, f and f^{-1} are continuous, where we are endowing the set $\mathbb{Q}$ with the **Hausdorff distance** (see Wikipedia) which simplifies in this case to the distance between $ABCD$ and $AB'CD'$ in $\mathbb{Q}$ is the **maximum** of the distances $|BB'|$ and $|DD'|$.

Now quadrilateral can be thought of as a point in the 4-cell. The various classes of quadrilaterals and their relations to one another can be visualized by finding equations and/or inequalities in the parameters which characterize them. These algebraic characterizations enable us to determine their dimension and whether they separate $\mathbb{P}$ into disjoint pieces. We can also draw some pictures.

Notations for the graph of the algebraic description of a class of quadrilaterals in $\mathbb{Q}$: *If $\mathcal{C}$ is a set of equations and/or inequalities in the parameters $t_i, i \in \{1,2,3,4\}$ then the points in $\mathbb{P}$ which satisfy them is $\mathbb{P}(\mathcal{C})$*, and $\mathbb{Q}(\mathcal{C})$ is the class of quadrilaterals $\mathbf{Q} \in \mathbb{Q}$ such that $f(\mathbf{Q}) \in \mathbb{P}(\mathcal{C})$.

So for example, $\mathbb{P}(t_2 = 1)$ is the graph of orthodiagonal quadrilaterals, $\mathbb{P}(t_3 = 1)$ is the graph of equidiagonal quadrilaterals, and $\mathbb{P}(t_2 = 1) \cap \mathbb{P}(t_3 = 1) = \mathbb{P}(t_2 = 1, t_3 = 1)$ is called the **midsquare quadrilaterals** in Josefsson's terminology[1],p.81. The class of the graph $\mathbb{P}(t_1 = 1)$ is the bimajor quadrilaterals. In any case, we see that *any class whose graph is contained in one or more of $\mathbb{P}(t_i = 1)$, $i \in \{1,2,3\}$ lies in the boundary of* $\mathbb{P}$.

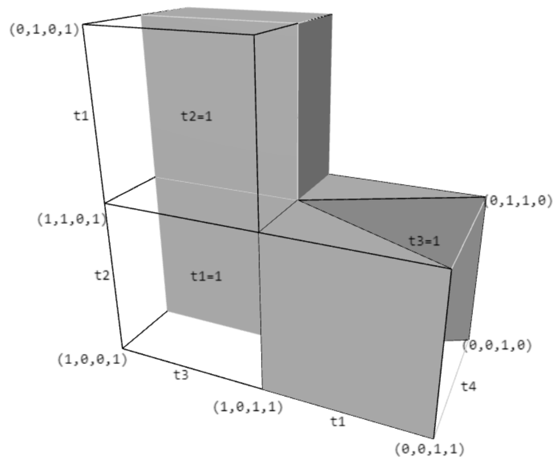

Figure 9: Graph of the boundary of $\mathbb{P}$

3.1. **Trapezoids.** There are two types of trapezoids in $\mathbb{Q}$, depending on whether AB and CD or BC and AD are parallel, These will be referred to these as **tall** and **short** trapezoids respectively.

You can see using alternate interior angles that $\triangle ABX$ and $\triangle CDX$ are similar for tall trapezoids and $\triangle BCX$ and $\triangle DAX$ are similar for short trapezoids. (see figure **Tall and Short trapezoids**)

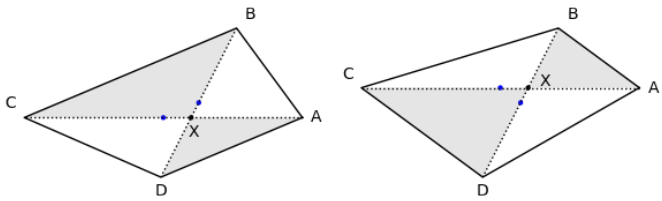

Figure 10: Short and Tall trapezoids

If $ABCD \in \mathbb{Q}$ is a tall trapezoid, then $\dfrac{|AX|}{|CX|} = \dfrac{|BX|}{|DX|}$, or $\dfrac{t_1}{2-t_1} = \dfrac{2\,t_3\,t_4}{2\,t_3\,(1-t_4)}$. This simplifies to $t_4 = \dfrac{t_1}{2}$, which gives us a description of the tall trapezoids in $\mathbb{Q}$ in terms of its parameters. Likewise, short trapezoids are characterized by the equation $t_4 = 1 - \dfrac{t_1}{2}$.

The remaining quadrilaterals (which form the vast majority) constitute 3 disjoint topological open 4-cells whose closures we will call **short, middle** and **tall quadrilaterals** respectively. This means that *in order to deform a short quadrilateral into a tall quadrilateral through an arc of quadrilaterals in* $\mathbb{Q}$*, it is necessary to pass through the middle quadrilaterals.*

We can also work out descriptions of the parallelograms, rectangles, rhombi, the square, equidiagonal quadrilaterals, and orthodiagonal quadrilaterals. From that, the dimension of each class is found and a diagram or picture of the class can be drawn in the parameter space. See the table **Algebraic description of some classes of quadrilaterals** at the end of this section.

The graphs which are 1, 2 or 3 dimensional can be visualized. For example the graph of the parallelograms, $\mathbb{P}(t_1 = 1, t_4 = .5)$ is dimension 2, in fact it is the 2-cell $\{1\} \times (0,1] \times (0,1] \times \{.5\} \subset \mathbb{P}$. It sits on the equidiagonal boundary $(t_1 = 1)$ of the quadrilaterals, and is the intersection of the tall and short trapezoids, each of which is a 3-cell having the parallelograms as a common face. The 3 or 4 dimensional graphs can be visualized by looking at **cross-sections** along any one of the 4 parameters. The graphs of the equidiagonal, orthodiagonal and bimajor quadrilaterals are examples of cross-sections of $\mathbb{P}$. Another example is the graph of the biminor quadrilaterals, $\mathbb{P}(t_4 = 1/2)$. This class **separates** $\mathbb{P}$ into two disjoint open pieces.

In the figure **The space of quadrilaterals** , each point represents a 2-cell of quadrilaterals, for example, the 2-cell of parallelograms sits above the point labeled parallelograms. The cross-section of biminor quadrilaterals bisects the middle quadrilaterals into two pieces. In fact, the mapping on the interior of $\mathbb{P}$ given by $f(t_1, t_2, t_3, t_4) = (t_1, t_2, t_3, 1 - t_4)$ is a reflection, that is, a homeomorphism which is its own inverse. It maps tall quadrilaterals to short and short to tall, with fixed point set the biminor quadrilaterals.

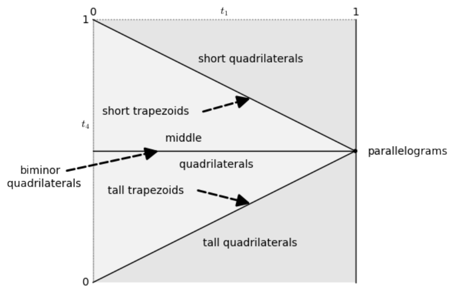

Figure 11: The space of quadrilaterals

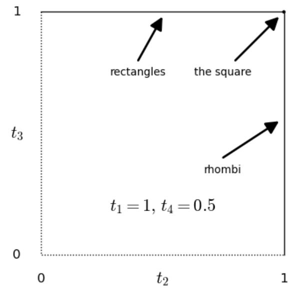

Figure 12: Parallelograms

The table below summarizes the classes of quadrilaterals, their algebraic characterizations, and their dimensions that we have discussed so far.

tall quadrilaterals:	$t_4 \leq \frac{1}{2}\, t_1$, dim 4
short quadrilaterals:	$t_4 \geq 1 - \frac{1}{2}\, t_1$, dim 4
middle quadrilaterals:	$\frac{1}{2}\, t_1 \leq t_4 \leq 1 - \frac{1}{2}\, t_1$, dim 4
interior quadrilaterals:	$t_1, t_2, t_3, t_4 \in (0, 1)$, dim 4
equidiagonal quadrilaterals:	$t_3 = 1$, dim 3
orthodiagonal quadrilaterals:	$t_2 = 1$, dim 3
bimajor quadrilaterals:	$t_1 = 1$, dim 3
biminor quadrilaterals:	$t_4 = \frac{1}{2}$, dim 3
tall trapezoids:	$t_4 = \frac{1}{2}\, t_1$, dim 3
short trapezoids:	$t_4 = 1 - \frac{1}{2}\, t_1$, dim 3
kites:	$t_2 = 1,\ t_4 = \frac{1}{2}$, dim 2
midsquare quadrilaterals:	$t_2 = 1,\ t_3 = 1$, dim 2
tall isosceles trapezoids:	$t_3 = 1,\ t_4 = \frac{1}{2} t_1$, dim 2
short isosceles trapezoids:	$t_3 = 1,\ t_4 = 1 - \frac{1}{2} t_1$, dim 2
parallelograms:	$t_1 = 1,\ t_4 = \frac{1}{2}$, dim 2
rectangles:	$t_1, t_3 = 1,\ t_4 = \frac{1}{2}$, dim 1
rhombi:	$t_1, t_2 = 1, t_4 = \frac{1}{2}$, dim 1
the square:	$t_1, t_2, t_3 = 1,\ t_4 = \frac{1}{2}$,dim 0

Table 1: Algebraic description of some classes of quadrilaterals

Note that in this nomenclature there are synonyms for many of the standard names of classes: parallelograms are bimajor biminor quadrilaterals, kites are orthogonal biminor quadrilaterals, rectangles are equidiagonal bimajor biminor quadrilaterals, rhombi are orthogonal bimajor biminor quadrilaterals, and the square is the only equidiagonal orthogonal bimajor biminor quadrilateral.

Next, let's examine some classes whose descriptions in the parameters are more complicated.

4. Circles and quadrilaterals.

There are at least three ways a circle can interact with a quadrilateral $ABCD$: (1) It might pass through A, B, C, D, or (2) it might be tangent to all four sides of $ABCD$ in which case its interior would be in the interior of $ABCD$, or (3) it might be tangent to all four lines AB,BC,CD, and DA and in the exterior of $ABCD$. This determines three classes of quadrilaterals: *Cyclic, Tangential, and Extangential* quadrilaterals.[1]. There are algebraic descriptions in terms of t_1, t_2, t_3, and t_4, which will enable us to determine the dimension and separation properties of these and other classes of quadrilaterals. It turns out that in the tangential and extangential cases, the equations are not useful for finding specific quadrilaterals, and we introduce a new set of parameters to do that.

4.1. **Cyclic quadrilaterals.** . We know that all angles inscribed in a circle and subtended by the same chord are equal, since an inscribed angle of a circle is equal to one half of the central angle subtended by the chord. As a consequence, if $ABCD \in \mathbb{Q}$ is cyclic, $\triangle AXB$ is similar with $\triangle DXC$. So $\dfrac{t_1}{2t_3(1-t_4)} = \dfrac{2t_3t_4}{2-t_1}$. Solve this for t_1 to get $t_1 = 1 - \sqrt{1-4t_3^2t_4(1-t_4)}$, that is, $\mathbb{P}(t_1 = 1-\sqrt{1-4t_3^2t_4(1-t_4)})$ is 3-dimensional. So the cyclic quadrilaterals form a 3-dimensional class in $\mathbb{Q}$.

Do the cyclic quadrilaterals separate the quadrilaterals? Yes, $\mathbb{Q}$ *is separated into two disjoint open sets by the cyclic quadrilaterals.* Here's a plausibility argument for this. First, for each $ABCD$ in $\mathbb{Q}$, let Circum($ABCD$) be the unique circle which passes through the three vertices A,B, and C of $ABCD$. Decompose $\mathbb{Q}$ into three disjoint sets: $\mathbb{C}in, \mathbb{C}, \mathbb{C}out$ where $ABCD \in \mathbb{C}in$, $\mathbb{C}$,or $\mathbb{C}out$ according to whether the vertex D lies inside, on, or outside Circum($ABCD$). So $\mathbb{C}$ is the class of cyclic quadrilaterals, and it is clear that if a quadrilateral from $\mathbb{C}in$ is deformed along an arc of quadrilaterals to one in $\mathcal{C}out$, then it must pass through $\mathbb{C}$.

We can visualize the graph of the cyclic quadrilaterals by drawing a typical t_2 = constant cross section of the graph. (See the drawing below). Then the graph is just the direct product of $(0, 1]$ with the cross-section. It is clear that the cyclic quadrilaterals separate the middle quadrilaterals into two disjoint open sets.

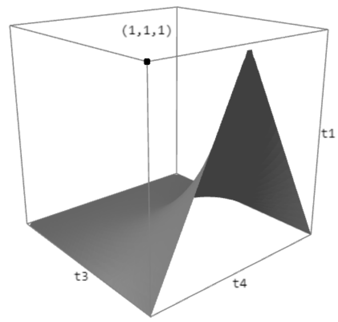

Figure 13: t_2 cross section of $\mathbb{P}$

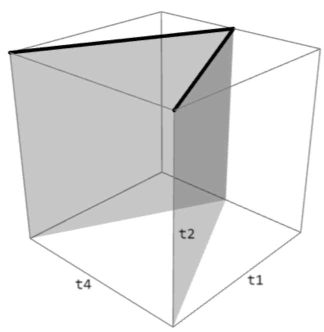

Figure 14: Graph of the isosceles trapezoids.

What about the intersection of the trapezoids with the cyclic quadrilaterals? Use a little algebra to show $\dfrac{t_1}{2t_3(1-t_4)} = \dfrac{2t_3t_4}{2-t_1}$ and ($t_1 = 2t_4$ or $t_1 = 1-2t_4$) implies $t_3 = 1$, so the intersection is the equidiagonal trapezoids, that is, the isosceles trapezoids.

Even though we can tell from the parameters of a quadrilateral if it is cyclic, we still don't know the center O or radius r of its circumcircle. But since the perpendicular bisector of any chord passes through the center, we do know that $O = (0, h)$ for some h, and from that $r = |O - A| = \sqrt{1 + h^2}$. But also $r = |O - B|$, yielding the equation $1 + h^2 = x^2 + (y - h)^2$ with $x = 1 - t_1 + 2t_3t_4c_2$ and $y = 2t_3t_4s_2$, $s_2 = \sin(t_2\pi/2)$, $c_2 = \cos(t_2\pi/2)$. Solve for h to get

$$h = \frac{x^2 + y^2 - 1}{2y} = \frac{(1 - t_1)^2 + (2t_3t_4)^2 + 4(1 - t_1)t_3t_4c_2 - 1}{4t_3t_4s_2}$$

.

4.2. **Tangential and extangential quadrilaterals.** The tangential quadrilaterals, the ones with an **inscribed circle**, are characterized by the simple equation $|AB| + |CD| = |BC| + |AD|$ (sums of opposite sides are equal). You can verify this equation by dropping the center of the incircle perpendicularly onto each side, and noting that each side of the equation decomposes into rearrangements of the same four numbers. The extangential quadrilaterals have a circle exterior to the quadrilateral which is tangent to all four extended sides. They were shown by Jacob Steiner in 1846 (see Wikipedia) to be characterized by the equations ($|AB|+|BC| = |CD|+|AD|$ or $|AB| + |AD| = |CB| + |CD|$). For extangential quadrilaterals in our model, this means that the exterior circle lies to the right of the major diagonal AC if $|AB| + |BC| = |CD| + |AD|$ and above the minor diagonal BD if $|AB| + |AD| = |CB| + |CD|$. We will term these **major extangential** and **minor extangential** quadrilaterals respectively.

An interesting observation about extangential quadrilaterals: Each extangential quadrilateral $ABCD$ in $\mathbb{Q}$ determines an ellipse $\frac{x^2}{(k/2)^2} + \frac{y^2}{(k/2)^2 - 1} = 1$ where $k = |AB| + |BC|$. A and C are the foci of the ellipse and B and D lie on the ellipse. When $|BD| < 2$, the quadrilateral is a major extangential quadrilateral. Not all pairs B and D yield and extangential quadrilateral. The x coordinate of B must be greater than the x coordinate of D, and B (and D) can't lie too far to the right (left): the exact ranges have not been worked out. If $|BD| > 2$, then the diagram should be reflected about the bisector of $\angle BXA$ and rescaled and the foci of the ellipse become B and D and the quadrilateral changes to a minor extangential quadrilateral.

There is a sagelet to play with the values of k and the x coordinates of B and D at *https://sagelets.cocalc.com/QuadSpace.html*.

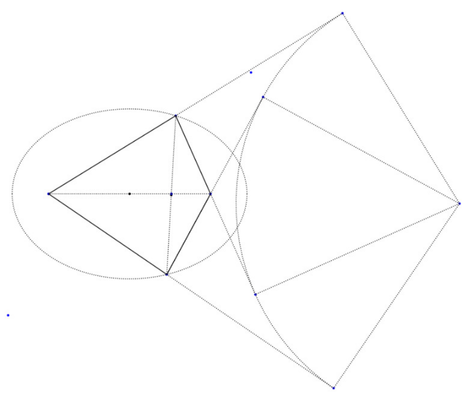

Figure 15: The ellipse of a major extangential quadrilateral

Each of the equations for the tangential and extangential quadrilaterals is a very ugly expression in the parameters t_1, t_2, t_3, t_4 equating the sum of two square roots with the sum of two other square roots. Since all four parameters are present in each equation, their graphs are not products, so we can't simply draw one cross section for each to see whether there is separation. But we do know that their graphs are 3 dimensional and we can draw their 2-dimensional cross sections in one of the parameters. Consider the $t_2 = .5$ cross sections of each equation drawn in the t_1, t_3, t_4 cube shown in the figures **View 1** and **View 2**. The cross sections for the first equation lie in the top half of the cube and converge **down** to $t_2 = 1, t_4 = 0.5$ and the cross sections for equation 2 le in the bottom half and converge **up** to $t_2 = 1, t_4 = .5$.

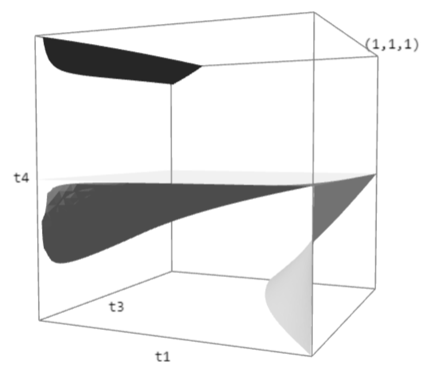

Figure 16: View 1

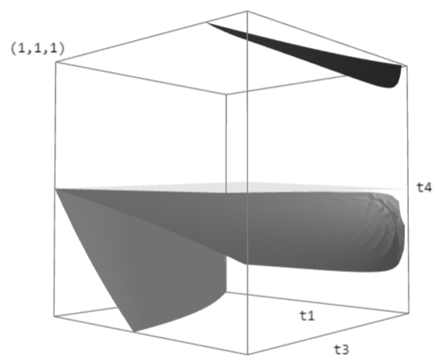

Figure 17: View 2

Do the tangential (respectively extangential) quadrilaterals separate $\mathbb{Q}$? A plausibility argument for separation analogous to the one for separation by cyclic quadrilaterals can be made here. For each $ABCD$ in $\mathbb{Q}$, let Tancir($ABCD$) be the unique circle which is tangent to the three sides DA,AB, and BC of $ABCD$. Decompose $\mathbb{Q}$ into three disjoint sets: 𝕋⊐⋉,𝕋∋⋉,𝕋⋊≊≈ where $ABCD \in$ 𝕋⊐⋉,𝕋∋⋉,or 𝕋⋊≊≈ according to whether the line CD meets Tancir($ABCD$) in 0, 1 , or 2 points. So 𝕋∋⋉ is the class of cyclic quadrilaterals, and it is clear that if a quadrilateral from 𝕋⊐⋉ is deformed along an arc of quadrilaterals to one in 𝕋⋊≊≈, then it must pass through 𝕋∋⋉. The argument for extangential quadrilateral is analogous. The figures **View 1** and **View 2** support these plausibility arguments. For any value of $t_2 < 1$, the graphs of the equations separate the the t_2 cross section of the t_1, t_3, t_4-cube into three disjoint open sets.

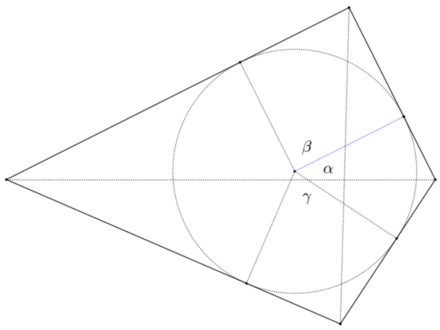

Figure 18: Tangential

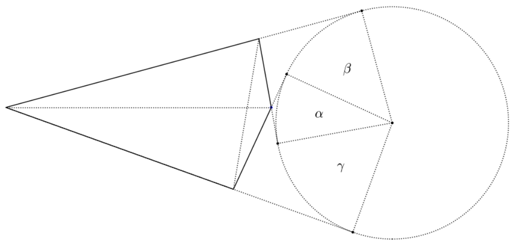

Figure 19: Major Extangential

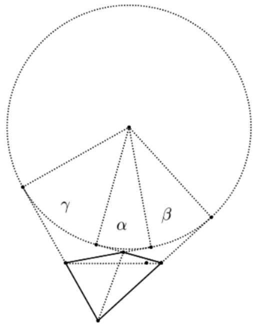

Figure 20: Minor Extangential

Since we can't solve (symbolically) either defining equation for any one of the variables, it is difficult to use the equations to determine a specific tangential or ex-tangential quadrilateral. One could resort to specifying 3 values and using numerical methods to determine the value of the 4th (if it exists). This method produces unreliable results for us. However, there is another parametrization which is much less sensitive, involving the three angles α, β, γ shown in figures 18, 19, and 20.

There is a sagelet implementing this parameterization at *https://sagelets.cocalc.com/QuadSpace.htr* With it, you can supply α, β, γ, and see it's picture and it's location in the model.

5. Questions

This model has proved helpful to the author in gaining new insights into the study of convex quadrilaterals. There are many more questions waiting to be found and answered in the subject. Here are some that occur to me.

5.1. **Quadrisections of convex quadrilaterals.** In [3], we made the conjecture that *if P is a convex polygon with $2n+1$ vertices, then it has at most $2n+1$ quadrisections.*. In efforts to prove this, we developed formulas for calculating the area of the upper right quadrant of each possible quadrisection. (A possible quadrisection is a pair of perpendicular lines each bisecting the area of the polygon. There is one for each angle between 0 and $\pi/2$ radians.) So far those efforts have failed to prove or disprove the conjecture. We didn't make the same conjecture for quadrisections of polygons with an even number of sides because the square has the property that each possible quadrisection *is* a quadrisection.

We stumbled onto a tall isosceles trapezoid, obtained by removing an isosceles triangle from the top of the unique isosceles triangle which two quadrisections[3], with exactly 5 quadrisections (See Figure 21). It has approximate parameters $t_1 = .48735$, $t_2 = .90947$, $t_3 = 1$, and $t_4 = .24369$. Figure 22 shows the graph of the **area function**, the area of the first quadrant minus one-fourth the area of the quadrilateral, as the possible quadrisection rotates through $\frac{\pi}{2}$ radians. Since the area of any polygon changes continuously (in fact the area function is differentiable) as its vertices change smoothly, we see that there is a small positive number ϵ such that if B' and D' are within ϵ of B and D respectively, then $AB'CD'$ also has 5 quadrisections. For this example, we can take $\epsilon = .00001$.

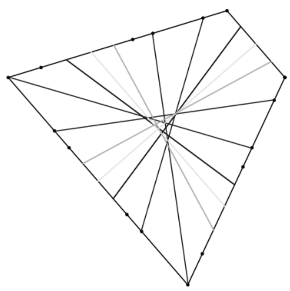

Figure 21: A trapezoid with 5 quadrisections.

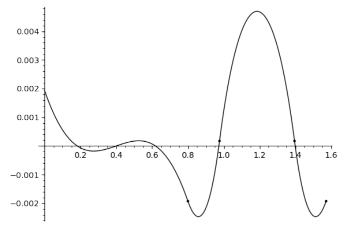

Figure 22: Its area function.

One can say the same about the degenerate quadrilateral $ABCD$ with $B = X$, $\angle AXD$ a right angle, and $DX = \frac{\sqrt{2}}{2}$ with parameters $t_1 = 1, t_2 = 1, t_3 = \sqrt{3}/2, t_4 = 0$, which is an equilateral triangle in the $t_4 = 0$ boundary of $\mathbb{P}$. Even though $ABCD$ is not in $\mathbb{P}$, there is a small positive number ϵ (.05 will do fine) such that each $AB'CD' \in \mathbb{P}$ with parameters $0 < t_1 - t_1' < \epsilon$, $0 < t_2 - t_2' < \epsilon$, $|t_3 - t_3'| < \epsilon$ and $0 < t_4 < \epsilon$ is in $\mathbb{Q}$ and has 3 quadrisections. Figure 23 shows an orthogonal kite with parameters $t_1 = 1$, $t_2 = 1$, $t_3 = 0.866$, and $t_4 = .05$.

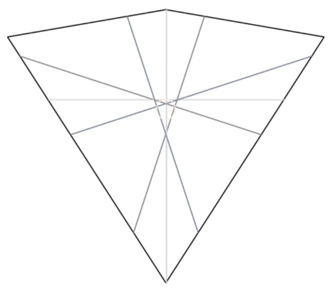

Figure 23: Kite with 3 quadrisections.

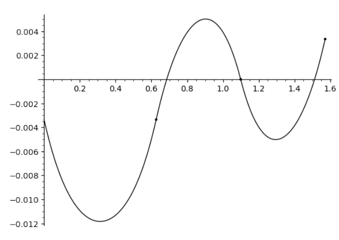

Figure 24: Its area function.

However, the same is not true about the square, which has infinitely many quadrisections. Indeed, any rhombus $ABCD$ other than the square has exactly one quadrisection, as can be seen from considering Figure 25. Work out that $y = (1 - x)h$ and $s = 1/(x/h + y)$. Then the area function is $A = h/2 - (x + sy)h/2 = \frac{(1 - h^2)x(1 - x)}{(1 - h^2)x + h^2}$. This is 0 at $x = 0$ or $x = 1$

when $0 < h < 1$ and 0 for **all** $x \in [0, 1]$ when $h = 1$ (ie when the rhombus is the square).

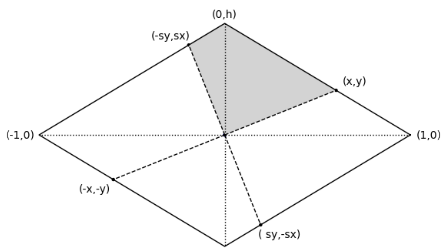

Figure 25: Nonquare rhombi have 1 quadrisection

Question 1: *Are there quadrilaterals with more than five but not infinitely many quadrisections?*

In the space of triangles, the vast majority of triangles have only one quadrisection except for a small open set about the equilateral triangle of triangles with 3 quadrisections whose boundary is an arc of triangles with 2 quadrisections.

Question 2: *Do the vast majority of quadrilaterals have only one quadrisection?*

Question 3: *Does every convex quadrilateral other than the square have only a finite number of quadrisections?*

5.2. **Compactifications of classes of quadrilaterals.** Our definition of convex quadrilateral does not include **degenerate quadrilaterals**, that is, quadrilaterals $ABCD$ where one or more of the vertices lies on the segment connecting two other vertices. However, when one or more of the parameters t_1, t_2, t_3, t_4 is 0 or $t_4 = 1$, the resulting quadrilateral $ABCD$ **is** degenerate. So the closed 4-cube $[0, 1] \times [0, 1] \times [0, 1] \times [0, 1] = [\mathbf{0}, \mathbf{1}]^{\mathbf{4}}$ is a compact space whose boundary contains one or more copies of each degenerate quadrilateral, and two copies of each (nonsquare) equidiagonal or orthodiagonal or bimajor quadrilateral. So it's not quite the parameter space of a model for the space of convex quadrilaterals including the degenerate ones. However, if we form the **quotient space** $[\mathbf{0}, \mathbf{1}]^{\mathbf{4}}/\sim$ obtained by identifying congruent quadrilaterals, we do get a model which includes the degenerate quadrilaterals.

Question: Describe this model. It is a 4-manifold with boundary. What does it look like? Draw the boundary.

On the way to answering the above question, it would be good to answer it for various types of quadrilaterals. For example, it is not hard to see that the compactification of the class of rectangles $\mathbb{P}(t_1 = 1, t_3 = 1, t_4 = 1/2)$ picks up just one degenerate quadrilateral $(1, 0, 1, 1/2) = f(ABCD)$ where $B = A$ and $D = C$, so the compactification of the class of rectangles is topologically a closed arc.

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[2] Gonzalez, Ahtziri and Lopez-Lopez, Jorge L., *Spaces of special quadrilaterals*, Bull. Aust. Math Soc. 100, pp. 155-167, 2019.

[3] Eberhart, Carl, *Revisiting the quadrisection problem of Jacob Bernoulli*, Forum Geometricorum, Vol 18, pp 7-16, 2018.

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