A model for the space of convex quadrilaterals

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Abstract. This paper describes a 'model' for the space of convex quadrilaterals, that is, a set $\mathbb{Q}$ of convex quadrilaterals in the plane which gives a 'cross-section' of the equivalence classes of similar convex quadrilaterals, in the sense that every convex quadrilateral is similar to exactly one member of $\mathbb{Q}$. It is also has the property that quadrilaterals which have nearly the same vertices are close to each other in the model in the Hausdorff metric. We also show that the model extends naturally to a model for all plane quadrilaterals including the simple quadrilaterals (those with the property that no two non-adjacent edges intersect) and the degenerate quadrilaterals (those with the property that at least one of B or D lies on the line through A and C).

A parameterization of the model with the 4-cell is defined and used to describe and investigate various classes of quadrilaterals. For example, the trapezoids form a 3-dimensional closed subset of the model which separates the model into 3 disjoint open sets, each homeomorphic with a 4-cell. we used **SageMath on CoCalc.com** to generate the figures and latex for this paper, and also to make the **Sage-Cell Interacts** to explore the model. Go to https://sagelets.cocalc.com/quad_index.html to find these.

The parameterization extends naturally to include all simple quadrilaterals, that is, quadrilaterals whose edges form a simple closed curve.

Quadrisections of convex polygons are discussed and questions are posed about the number of quadrisections of a convex quadrilateral.

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1. Introduction

Recall that a *convex quadrilateral* is a plane polygon with 4 vertices whose diagonals intersect in interior interior. There are many different kinds of convex quadrilaterals: *trapezoid*, *parallelogram*, *rhombus*, *rectangle*, *kite*, *cyclic quadrilateral* and many others. Over the centuries, there have been numerous classification schemes proposed. Martin Josefsson [1] has given a good summary of these and has proposed another interesting classification. Also, Ahtziri Gonzalez and Jorge L. Lopez-Lopez have established some of our results using an alternative model (see [2],[3]).

We will describe a 'model' for the space of convex quadrilaterals, that is, a set $\mathbb{Q}$ of convex quadrilaterals in the plane which gives a 'cross section' of the equivalence classes of similar convex quadrilaterals in the sense that every convex quadrilateral is similar to exactly one member of $\mathbb{Q}$. This model also has the property that quadrilaterals which have nearly the same vertices are close to each other. However, it unavoidably also has nearly congruent quadrilaterals which are far apart.

We modeled the space of triangles in [4]. By scaling and a Euclidean motion, we can place each triangle so that the its vertices are $A = (1, 0)$, $B = (x, y)$, and $C = (-1, 0)$, where $x \geq 0$, $y > 0$, and $(x+1)^2 + y^2 \leq 4$, that is, B is in first quadrant above the x-axis and $|BC| \leq 2$, where $|BC|$ is the Euclidean distance between points B and C. That model is 2 dimensional and is a disk with a closed arc removed from the boundary (see Fig. 1). We also investigated how the various types of triangles were situated in the model.

For example, the boundary of the model consists of an open interval of isosceles triangles with the equilateral triangle at the midpoint with the tall isosceles triangles to the right and the short isosceles triangles to the left. Also, the right triangles forms an arc (half-open) ending at the isosceles right triangle and separating the model into two components, one being all triangles with an obtuse angle and the other being all triangles with no obtuse angle.

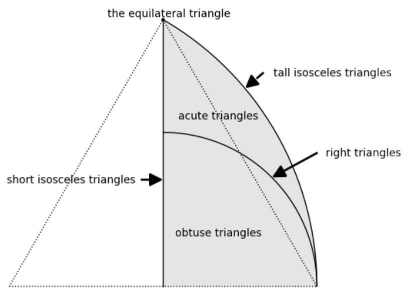


Fig. 1: Space of triangles.

We were primarily interested in identifying the triangles with one, two or three *quadrisections* (perpendicular segments dividing a triangle into four equal areas), and found that almost all triangles have only one quadrisection save a small open (in the model) set of triangles with three quadrisections about the equilateral triangle. The boundary of this is a closed arc separating the triangle with three quadrisections from those with only one quadrisection, starting at the only isosceles triangle with two quadrisections and consisting of scalene triangles with two quadrisections except for the other endpoint which is an isosceles triangle with one quadrisection.

2. Description of the Model

Any model for the convex quadrilaterals is 4 dimensional. Using elementary Cartesian geometry, we can fix endpoints of a diagonal and let the other two vertices roam subject to some conditions that require the quadrilateral to be convex and not degenerate to a triangle or segment. Also we want every quadrilateral to be similar to exactly one quadrilateral in our model.

To start, we will build our model for the convex quadrilaterals from the set $\overline{\mathbb{Q}}$ of quadrilaterals $ABCD$ defined next. Then we will identify congruent members of $\overline{\mathbb{Q}}$ and devise rules for which ones to keep in $\mathbb{Q}$.

Definition 2.1. *$\overline{\mathbb{Q}}$ consists of those quadrilaterals $\mathcal{Q}$ with vertices $ABCD$ labeled counterclockwise so that $A = (1, 0)$, $C = (-1, 0)$ and the diagonal BD has length $|BD|$ no more than 2 and crosses the x-axis at a point X between A and C so that $0 < |AX| \leq |CX|$ and $\angle BXA$ is not obtuse.*

We call AC and BD the **major** and **minor** diagonals of $ABCD$.

Theorem 2.1. *Each convex quadrilateral $\mathcal{P}$ is similar with at least one member of $\overline{\mathbb{Q}}$.*

Choose a diagonal of $\mathcal{P}$ of maximum length. Now scale, translate and rotate $\mathcal{P}$ onto a quadrilateral $\mathcal{Q}$ endpoints of that diagonal are $(1,0)$ and $(-1,0)$. Label the endpoints of the other diagonal B and D, so that $\mathcal{Q} = ABCD$ is labeled in counterclockwise order. $\mathcal{Q}$ is similar to the original $\mathcal{P}$. Let X be the intersection of BD and AC. Then $0 < |AX| \leq |CX|$ or $0 < |CX| < |AX|$ and $\angle BXA$ is obtuse or not obtuse. If $\angle BXA$ is not obtuse and $0 < |AX| \leq |CX|$, then $\mathcal{Q} \in \overline{\mathbb{Q}}$. If $\angle BXA$ is obtuse and $0 < |CX| < |AX|$, then reflect $ABCD$ about the y-axis to put it in $\overline{\mathbb{Q}}$. If $\angle BXA$ is obtuse and $0 < |AX| \leq |CX|$, then reflect $ABCD$ about the x-axis to put it in $\overline{\mathbb{Q}}$. If $\angle BXA$ is not obtuse and $0 < |CX| < |AX|$, then reflect $ABCD$ about the x-axis then the y-axis to put it in $\overline{\mathbb{Q}}$. So $\mathcal{P}$ is similar to at least one member of $\overline{\mathbb{Q}}$. ■

In the following discussion, $\mathcal{Q}$ always denotes a member of Q and its vertices are $ABCD$, labeled counterclockwise with $A = (1,0)$, $C = (-1,0)$, X is the intersection of AC and BD. $\mathcal{Q}_x$, $\mathcal{Q}_y$, and $\mathcal{Q}_{xy}$ are the quadrilaterals obtained by reflecting $\mathcal{Q}$ about the x-axis, the y-axis, and the x-axis then y-axis respectively, not all necessarily distinct. If $|BD| = 2$, then by theorem 2.1, one of the four additional candidates for membership in $\overline{\mathbb{Q}}$ must lie in $\overline{\mathbb{Q}}$. So in some cases, there may be congruent copies of $\mathcal{Q}$ in $\overline{\mathbb{Q}}$. We need to find them and devise rules to eliminate all but one of them from $\overline{\mathbb{Q}}$.

Some terminology to facilitate sorting this out is useful. If the diagonals of a quadrilateral are orthogonal, then it is an **orthodiagonal** quadrilateral. If the diagonals are the same length, then it is **equidiagonal**. These are standard terms used for example in [1].

If the diagonals are different lengths, we call the longer diagonal the **major diagonal** and the shorter one the **minor diagonal**. If a quadrilateral $\mathcal{Q} \in \overline{\mathbb{Q}}$, then AC is the major diagonal and BD is the minor diagonal even if $|BD| = 2$.

If the major diagonal is bisected by the minor diagonal, then the quadrilateral is **bimajor**. If the minor diagonal is bisected by the major diagonal, then it is **biminor**. I believe these terms to be new.

Combinations of these terms characterize some common classes of quadrilaterals. An orthodiagonal, equidiagonal, bimajor, biminor quadrilateral is a square. Rhombi are orthodiagonal, bimajor and biminor quadrilaterals. Kites are orthodiagonal, biminor quadrilaterals. Rectangles are equidiagonal, bimajor, biminor quadrilaterals, and parallelograms are bimajor, biminor quadrilaterals.

A natural generalization of the property **bimajor and biminor** might be called **equiproportional**, the diagonals divide each other in the same proportion (i.e. $\frac{|AX|}{|CX|} = \frac{|BX|}{|DX|}$ or $\frac{|AX|}{|CX|} = \frac{|DX|}{|BX|}$). But that is equivalent to $AB \parallel CD$ or $AD \parallel BC$, which defines a trapezoid, so we won't coin it.

Now we can pin down exactly when $\overline{\mathbb{Q}}$ contains a duplicate copy $\mathcal{Q}'$ of a member $\mathcal{Q}$.

First, if $\mathcal{Q}$ is not equidiagonal, then there are only three candidates for $\mathcal{Q}'$, namely $\mathcal{Q}_x$, and $\mathcal{Q}_{xy}$, and $\mathcal{Q}_y$.

And if $\mathcal{Q}$ is equidiagonal, there are another four candidates obtained from moving BD onto AC by a rigid motion. All seven of these candidates are equal to $\mathcal{Q}$ if $\mathcal{Q}$ is the square with $B = (0, 1)$ and $D = (0, -1)$. It's also easy to see that any non equidiagonal kite has only one similar member in $\overline{\mathbb{Q}}$. The next theorem tells us that 'most' quadrilaterals have only one similar member in Q.

Theorem 2.2. *Suppose $\mathcal{Q} = ABCD \in \overline{\mathbb{Q}}$. Then one of $\mathcal{Q}_x$, $\mathcal{Q}_y$, and $\mathcal{Q}_{xy}$ is a different member of $\overline{\mathbb{Q}}$ if and only if $\mathcal{Q}$ is (i) orthodiagonal and not biminor, or (ii) bimajor and not biminor.*

Hence, a quadrilateral which is not equidiagonal is similar to just one member of $\overline{\mathbb{Q}}$ unless it is orthodiagonal but not biminor or it is bimajor but not biminor, in which cases it is similar to exactly two members of $\overline{\mathbb{Q}}$.

Proof. $\mathcal{Q}_y \in \overline{\mathbb{Q}}$ and $\mathcal{Q}_y \neq \mathcal{Q}$ if and only if $X = (0, 0)$ and BD is vertical if and only if $\mathcal{Q}_y = \mathcal{Q}$, a contradiction.

$\mathcal{Q}_x \in \overline{\mathbb{Q}}$ and $\mathcal{Q}_x \neq \mathcal{Q}$ if and only if BD is vertical and $|BX| \neq |DX|$, if and only if $\mathcal{Q}$ is orthodiagonal and not biminor.

$\mathcal{Q}_{xy} \in \overline{\mathbb{Q}}$ and $\mathcal{Q}_{xy} \neq \mathcal{Q}$ if and only if $X = (0, 0)$ is vertical and $|BX| \neq |DX|$, if and only if $\mathcal{Q}$ is bimajor and not biminor. ■

For equidiagonal quadrilaterals we find that is some cases there are two, three, or even four elements of $\overline{\mathbb{Q}}$ which are similar with it.

Here is a useful lemma.

Lemma 2.1. *An equidiagonal $\mathcal{Q} = ABCD \in \overline{\mathbb{Q}}$ is a trapezoid if and only if $|AX| = |BX|$ or $|AX| = |DX|$.*

We know $\mathcal{Q}$ is a trapezoid if and only if $\dfrac{|AX|}{|CX|} = \dfrac{|BX|}{|DX|}$ or $\dfrac{|AX|}{|CX|} = \dfrac{|DX|}{|BX|}$. Since $\mathcal{Q}$ is equidiagonal, $\triangle ABX$ is similar with $\triangle CDX$ or $\triangle AXD$ is similar with $\triangle CXB$ if and only if $\dfrac{|AX|}{|CX|} = \dfrac{|BX|}{|DX|}$ or $\dfrac{|AX|}{|CX|} = \dfrac{|DX|}{|BX|}$ if and only if $\dfrac{|AX|}{2 - |AX|} = \dfrac{|BX|}{2 - |BX|}$ or $\dfrac{|AX|}{2 - |AX|} = \dfrac{|DX|}{2 - |DX|}$ if and only if $|AX| = |BX|$ or $|AX| = |DX|$. ■

Theorem 2.3. *Suppose $\mathcal{Q} = ABCD \in \overline{\mathbb{Q}}$ is equidiagonal. Then*

(i) $\mathcal{Q}$ is congruent with three other members of $\overline{\mathbb{Q}}$ if and only if $\mathcal{Q}$ is orthodiagonal, not bimajor, not biminor and not a trapezoid.

(ii) $\mathcal{Q}$ is congruent with two other members of $\overline{\mathbb{Q}}$ if and only if $\mathcal{Q}$ is orthodiagonal, and bimajor or biminor but not both.

(iii) $\mathcal{Q}$ is congruent with one other member of $\overline{\mathbb{Q}}$ if and only if $\mathcal{Q}$ (a) is not a trapezoid and not orthodiagonal or (b) is an orthodigonal trapezoid which is not the square .

(iv) $\mathcal{Q}$ is congruent with no other member of $\overline{\mathbb{Q}}$ if and only if $\mathcal{Q}$ is (i) a trapezoid, (ii) orthodiagonal and biminor (i.e. a kite), or (iii) not equidiagonal, not orthodiagonal, and not bimajor.

Proof. (i) $\mathcal{Q}$ is orthodiagonal, not bimajor, not biminor and not a trapezoid if and only if BD is vertical and all four of $|AX|$, $|BX|$, $|CX|$, and $|DX|$ are distinct. So $\mathcal{Q} \neq \mathcal{Q}_x \in \overline{\mathbb{Q}}$. $|BX| < 1$ or $|DX| < 1$. We can assume $|BX| < 1$. Then the $\mathcal{Q}'$ obtained by reflecting $\mathcal{Q}$ about the bisector of $\angle BXA$ followed by a horizontal translation to place B on $(1, 0)$ and D on $(-1, 0)$ is in $\overline{\mathbb{Q}}$ and different from $\mathcal{Q}$ and $\mathcal{Q}_x$. Also $\mathcal{Q}'_x$ is in $\overline{\mathbb{Q}}$ and different from $\mathcal{Q}$, $\mathcal{Q}_x$ and $\mathcal{Q}'$.

(ii) Assume $\mathcal{Q}$ is biminor but not bimajor. Then $\mathcal{Q}_x = \mathcal{Q}$. It is not a trapezoid so $|AX|$, $|BX|$, and $|DX|$ are distinct. Then reflect $\mathcal{Q}$ about the bisector of $\angle BXA$ and perform a horizontal translation to place B on $(1, 0)$ and D on $(-1, 0)$ to obtain $\mathcal{Q}'$ different from $\mathcal{Q}$ and $\mathcal{Q}'_x$.

(iii) If $\mathcal{Q}$ is not orthodiagonal, $\angle BXA$ is acute, so $\mathcal{Q}_x \notin \overline{\mathbb{Q}}$. But $\mathcal{Q}'$ is obtained the same way as in (i), and $\mathcal{Q}'_x \notin \overline{\mathbb{Q}}$. If $\mathcal{Q}$ is an orthodiagonal trapezoid other than the square, then $|AX| < 1$ is equal to $|BX|$ or $|DX|$, and the $\mathcal{Q}'$ obtained as in (1) is in $\overline{\mathbb{Q}}$ but distinct from $\mathcal{Q}$.

(iv) If $\mathcal{Q}$ is a trapezoid with $\angle BXA$ acute, then none of $\mathcal{Q}_x$, $\mathcal{Q}_y$, and $\mathcal{Q}_xy$ are in $\overline{\mathbb{Q}}$. Also reflection of $\mathcal{Q}$ about the bisector of $\angle BXA$ carries $\mathcal{Q}$ onto itself.

Note all the rectangles are included here. And the square fits into this category also. ■

What does this theorem tell us? That the only quadrilaterals which are similar to more than one member of $\overline{\mathbb{Q}}$ are orthodiagonal, equidiagonal or bimajor. These lie in the **boundary** of $\overline{\mathbb{Q}}$, that is, an arbitralily small change in the position of the vertex B can move $ABCD$ out of $\overline{\mathbb{Q}}$. So the vast majority of quadrilaterals are similar to exactly one member of $\overline{\mathbb{Q}}$.

In order to satisfy our requirement that $\mathbb{Q}$ contain a single member similar with a given quadrilateral, we must adopt rules for eliminating all but one copy from $\overline{\mathbb{Q}}$.

Rules for membership in $\mathbb{Q}$

(1) If $\mathcal{Q} = ABCD \in \overline{\mathbb{Q}}$ is not equidiagonal, then
if $\mathcal{Q}$ is (i) orthodiagonal but not biminor or (ii) bimajor but not biminor, then there is another member of $\overline{\mathbb{Q}}$ congruent with $\mathcal{Q}$. One has $|BX| < |DX|$ and the other $|BX| > |DX|$. Put $\mathcal{Q}$ in $\mathbb{Q}$ if $|BX| < |DX|$ but not the other one.

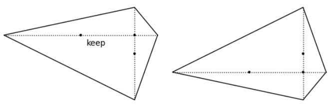


Fig. 2: Orthodiagonal not biminor

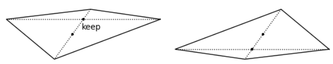


Fig. 3: Bimajor not biminor

(2) If $\mathcal{Q} = ABCD \in \overline{\mathbb{Q}}$ is equidiagonal, then
(i) if $\mathcal{Q}$ is orthodiagonal, not bimajor, not biminor and not a trapezoid, then there are three other members of $\overline{\mathbb{Q}}$ congruent with $\mathcal{Q}$. Eliminate $\mathcal{Q}$ from $\overline{\mathbb{Q}}$ if $\min\{|BX|, |DX|\} < |AX|$ or $|DX| < |BX|$ (thus, we keep $ABCD \in \mathbb{Q}$ if $|AX| < |BX| < |DX|$),

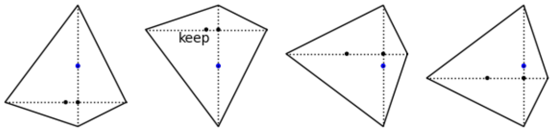


Fig. 4: Orthodiagonal, not bimajor, not biminor and not a trapezoid

(ii) or if $\mathcal{Q}$ is orthodiagonal and biminor or bimajor but not both, then there are two other members of $\overline{\mathbb{Q}}$ congruent with $\mathcal{Q}$. Eliminate

$\mathcal{Q}$ from $\mathbb{Q}$ if $|AX| < 1$ or $|DX| < |BX|$ (thus, we keep $ABCD \in \mathbb{Q}$ if $|BX| < |DX|$ and $|AX| = 1$),

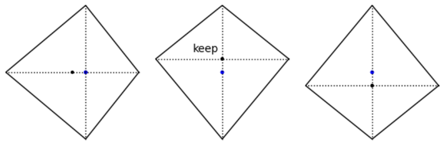


Fig. 5: Orthodiagonal, and bimajor or biminor but not both

(iii) or if $\mathcal{Q}$ is (a) not orthodiagonal and not trapezoid, then there is another member of $\overline{\mathbb{Q}}$ congruent with $\mathcal{Q}$. Eliminate $\mathcal{Q}$ if $|AX| > \min\{|BX|, |DX|\}$ (thus keep $\mathcal{Q}$ if $|AX| < \min\{|BX|, |DX|\}$) or (b) is an orthodigonal trapezoid which is not the square, then there is another member of $\overline{\mathbb{Q}}$ congruent with $\mathcal{Q}$. Eliminate $\mathcal{Q}$ if $|DX| < |BX|$ (thus keep $ABCD \in \mathbb{Q}$ if $|BX| < |DX|$).

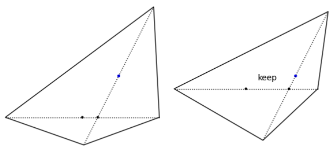


Fig. 6: Not orthodiagonal or trapezoid

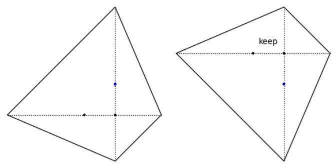


Fig. 7: Orthodiagonal trapezoid

Notice that the rules for elimination always keep $ABCD \in \mathbb{Q}$ when $|BX| < |DX|$, with the exceptions of when $ABCD$ is equidiagonal and not orthodiagonal and not a trapezoid and when $ABCD$ is an equidiagonal, orthodiagonal trapezoid (ie an isosceles trapezoid).

At this point, the description of the model $\mathbb{Q}$ is complete. Now we will **parameterize** $\mathbb{Q}$ and use the **parameter space** to help visualize the various classes of quadrilaterals and their relation to each other.

3. Parameterizing the Model.

We could use the coordinates of $B = (x, y)$ and $D = (y, w)$, to parameterize $ABCD$, but these are very inconvenient parameters to work with. As subspaces of $\mathbb{R}^4$, $\overline{\mathbb{Q}}$ and $\mathbb{Q}$ are difficult to visualize. Ordinarily to visualize a subset of $\mathbb{R}^4$, we would graph the three, two and one dimensional cross sections obtained by fixing the value of one, two or three of the variables x, y, z, w. But then you have to work out the ranges for the remaining variables, and that is difficult and uninformative.

A much more useful set of parameters for a member $\mathcal{Q} = ABCD$ of $\overline{\mathbb{Q}}$ comes from the major and minor diagonals AC and BD of $\mathcal{Q}$ and their intersection point $X = (r, 0)$, which can be written as a function of (x, y, z, w): $r = x - \dfrac{y}{y - w}(x - z)$. In fact, these parameters make perfect sense for members of $\overline{\mathbb{Q}}$.

Definition 3.1. *For each $\mathcal{Q} = ABCD = (x, y, z, w) \in \overline{\mathbb{Q}}$, define a unique 4-tuple $f(\mathcal{Q}) = f(x, y, z, w) = (t_1, t_2, t_3, t_4) \in [0, 1]^4$ as follows:*

$t_1 = \dfrac{2\,|AX|}{|AC|} = 1 - r = 1 - x + \dfrac{y}{y - w}(x - z)$. *So $t_1 \in (0, 1]$ when X is in the right hand half of AC.*

$t_2 = \dfrac{|BP|}{|BD|} = \dfrac{y-w}{\sqrt{(x-z)^2+(y-w)^2}}$, *where* $P = (x, w)$. *So* $t_2 \in (0, 1]$ *is the sine of* $\angle AXB$, *and* $\sqrt{1-t_2^2}$ *is the cosine of* $\angle AXB$.

$t_3 = \frac{1}{2}|BD| = \frac{1}{2}\sqrt{(x-z)^2+(y-w)^2}$. *So since* $2\,t_3 = |BD| \leq |AC| = 2$, $t_3 \in (0, 1]$.

$t_4 = \dfrac{|BX|}{|BD|} = \dfrac{y}{y-w}$, *so* $t_4 \in (0, 1)$. *Note that* $|BX| = 2\,t_3\,t_4$ *and* $|DX| = 2\,t_3\,(1-t_4)$.

We call $\overline{\mathbb{P}} = f(\overline{\mathbb{Q}})$ *the* ***Parameter Space*** *of* $\overline{\mathbb{Q}}$ *and* $\mathbb{P} = f(\mathbb{Q})$ *the* ***Parameter Space*** *of* $\mathbb{Q}$.[1]

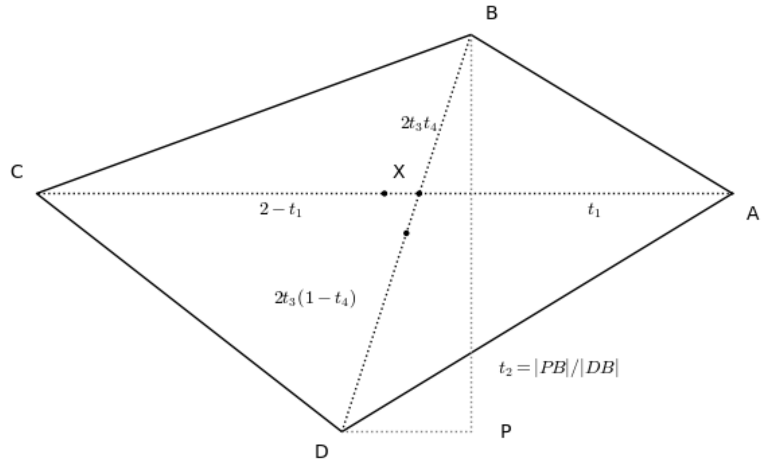


Fig. 8: Parameters for $ABCD$.

Note that $\mathcal{Q}$ is bimajor if and only if $t_1 = 1$, orthodiagonal if and only if $t_2 = 1$, equidiagonal if and only if $t_3 = 1$ and biminor if and only if $t_4 = .5$.

Theorem 3.1. *The mapping* $f : \overline{\mathbb{Q}} \to \overline{\mathbb{P}}$ *is a homeomorphism, carrying* $\mathbb{Q}$ *onto* $\mathbb{P}$.

Proof. The coordinate functions of f are continuous where defined since they are algebraic.

To show f is 1-1, suppose $f(x, y, z, w) = (t_1, t_2, t_3, t_4) = f(x_1, y_1, z_1, w_1)$. Then

(1) $2t_3 = \sqrt{(x-z)^2+(y-w)^2} = \sqrt{(x_1-z_1)^2+(y_1-w_1)^2}$.

(2) $t_2 = \dfrac{y-w}{\sqrt{(x-z)^2+(y-w)^2}} = \dfrac{y_1-w_1}{\sqrt{(x-z)^2+(y-w)^2}}$ using (1), so $y - w = y_1 - w_1$.

(3) $t_4 = \dfrac{y}{y-w} = \dfrac{y_1}{y_1-w_1}$, so $y = y_1$ using (2), and therefore $w = w_1$.

(4) By (1) and (2), $x - z = x_1 - z_1$.

(5) $t_1 = 1 - x + \dfrac{y}{y-w}(x-z) = 1 - x_1 + \dfrac{y}{y-w}(x-z)$ using (3) and (4), and so $x = x_1$ and therefore $z = z_1$.

[1] In the sagelets located at *https://sagelets.cocalc.com/quad_index.html*, we rescale the parameter t_2 to the ratio of $\angle BXA$ (measured in degrees) to 90, so that $t_2 = 1/3$ corresponds to $\angle BXA = 30^o$, etc.

Therefore f is one-to-one onto $f(\overline{\mathbb{Q}}) = \overline{\mathbb{P}}$

Now define $g : \overline{\mathbb{P}} \to \overline{\mathbb{Q}}$ by $g(t_1, t_2, t_3, t_4) = (x, y, z, w)$ where

$x = 1 - t_1 + 2\,t_3\,t_4\,\sqrt{1 - t_2^2}$ $\quad y = 2\,t_3\,t_4\,t_2$

$z = 1 - t_1 - 2\,t_3\,(1 - t_4)\,\sqrt{1 - t_2^2}$ $\quad w = -2\,t_3\,(1 - t_4)\,t_2$

g is continuous since its coordinate functions are continuous. We claim that $fg(t_1, t_2, t_3, t_4) = f(x, y, z, w) = (t_1, t_2, t_3, t_4)$. To do this it suffices to solve the above equations for t_1, t_2, t_3, t_4.

(1) From $x - z = 2\,t_3\,\sqrt{1 - t_2^2}$, and $y - w = 2\,t_3\,t_2$, get $(x - z)^2 + (y - w)^2 = 4\,t_3^2$. Thus $t_3 = \frac{1}{2}\sqrt{(x - z)^2 + (y - w)^2}$.

(2) From $y - w = 2\,t_3\,t_2$ and (1) get $t_2 = \dfrac{y - w}{\sqrt{(x - z)^2 + (y - w)^2}}$.

(3) From $y = 2\,t_3\,t_4\,t_2$ and (1) and (2), get $y = t_4\,(y - w)$ or $t_4 = \dfrac{y}{y - w}$.

(4) From $x = 1 - t_1 + 2\,t_3\,t_4\sqrt{1 - t_2^2}$ and (1), (2), and (3), get $t_1 = 1 - x + \sqrt{(x - z)^2 + (y - w)^2}\,\dfrac{y}{y - w}\sqrt{1 - \dfrac{(y - w)^2}{(x - z)^2 + (y - w)^2}}$, which simplifies to $t_1 = 1 - x + \dfrac{y}{y - w}(x - z)$

This completes the proof that $\overline{\mathbb{Q}}$ and $\overline{\mathbb{P}}$ are homeomorphic via f. Since $\mathbb{P} = f(\mathbb{Q})$ by definition, f carries $\mathbb{Q}$ homeomorphically onto $\mathbb{P}$. ■

So now we know that $\overline{\mathbb{Q}}$ and $\mathbb{Q}$ are homeomorphic with $\overline{\mathbb{P}}$ and $\mathbb{P}$ respectively, but what are $\overline{\mathbb{P}}$ and $\mathbb{P}$? The next two theorems show that they are particular subsets of the unit 4-cell $\mathbb{I}^4 = [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$

Some terminology for the 4-cell: The open 4-cell $\mathring{\mathbb{I}}^4 = (0, 1) \times (0, 1) \times (0, 1) \times (0, 1)$ is the **interior** of $\mathbb{I}^4$. The **boundary** of $\mathbb{I}^4$ consists of eight 3-cells, called **faces**, $\mathbb{I}^3_{i,j} = \{(t_1, t_2, t_3, t_4) \in \mathbb{I}^4 | t_i = j\}$, $i \in \{1, 2, 3, 4\}$, $j \in \{0, 1\}$.

Theorem 3.2.

(1) *$\overline{\mathbb{P}}$ consists of the 4-cell with five faces removed: $\mathbb{I}^3_{i,0}$ for $i \in \{1, 2, 3, 4\}$ and $\mathbb{I}^3_{4,1}$, that is $\overline{\mathbb{P}} = (0, 1] \times (0, 1] \times (0, 1] \times (0, 1)$.*

(2) *The interior of $\overline{\mathbb{P}}$ is $\mathring{\mathbb{I}}^4$, and the boundary of $\overline{\mathbb{P}}$ is $\overline{\mathbb{P}} \cap (\mathbb{I}^3_{1,1} \cup \mathbb{I}^3_{2,1} \cup \mathbb{I}^3_{3,1})$, that is, the set of points in $\overline{\mathbb{P}}$ with at least one the its first three coordinates 1.*

Proof.

(1) It follows from the definition of the parameters that $\overline{\mathbb{P}} \subseteq (0, 1] \times (0, 1] \times (0, 1] \times (0, 1)$. Conversly, suppose $(t_1, t_2, t_3, t_4) \in (0, 1] \times (0, 1] \times (0, 1] \times (0, 1)$. Then let $\mathcal{Q} = ABCD$ where $A = (1, 0)$, $C = (-1, 0)$, and $B = (x, y) = (t_1, 0) + 2\,t_3\,t_4\,U$, and $D = (z, w) = (t_1, 0) - 2\,t_3\,(t_4\,U$, where $U = (\cos(\theta), \sin(\theta))$ is the unit vector in the direction $\theta = \arcsin(t_2\frac{\pi}{2})$. Straightforward calculation shows that $ABCD \in \overline{\mathbb{Q}}$ and therefore $f(ABCD) \in \overline{\mathbb{P}}$.

(2) $\mathring{\mathbb{I}}^4$ is open in $\overline{\mathbb{P}}$, since it is open in $\mathbb{R}^4$. $\overline{\mathbb{P}} \setminus \mathring{\mathbb{I}}^4 = \overline{\mathbb{P}} \cap (\mathbb{I}^3_{1,1} \cup \mathbb{I}^3_{2,1} \cup \mathbb{I}^3_{3,1})$ contains no interior point of $\overline{\mathbb{P}}$.

■

Now we can determine $\mathbb{P}$ using Theorems 2.3 and 3.2.

Corollary 3.1. $\mathbb{P} = \mathring{\mathbb{I}}^4 \cup \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3 \cup \mathbb{B}_4$, *where* $\mathbb{B}_1 = \{1\} \times (0,1] \times (0,1] \times (0,1/2]$, $\mathbb{B}_2 = (0,1] \times \{1\} \times (0,1] \times (0,1/2]$, $\mathbb{B}_3 = \{(t_1,1,1,t_4) \mid 0 < t_1 < 1$ *and* $0 < t_4 \leq t_1/2\}$, *and* $\mathbb{B}_4 = \{(t_1,t_2,1,t_4)| \; 0 < t_1,t_2 < 1, 0 < t_4 \leq t_1/2$ *or* $t_4 > 1 - t_1/2\}$

Proof. It follows from Theorem 3.2 (2) that $\mathring{\mathbb{I}}^4 \subset \mathbb{P}$. It follows from Theorem 2.3, the Rules for membership in $\mathbb{Q}$ and Theorem 3.2 that $\mathbb{P} \setminus \mathring{\mathbb{I}}^4 = \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3 \cup \mathbb{B}_4$. ■

4. Graphs of classes of quadrilaterals

Now that a quadrilateral is identified with a point in the 4-cell, the various classes of quadrilaterals and their relations to one another can be visualized by finding equations and/or inequalities in the parameters t_1, t_2, t_3, t_4 which characterize them. These algebraic characterizations make it simple to determine their dimension and whether they separate $\mathbb{P}$ into disjoint pieces. We can also draw some pictures.

Notations for the graph of the algebraic description of a class of quadrilaterals in $\mathbb{Q}$: *If $\mathcal{C}$ is a set of equations and/or inequalities in the parameters $t_i, i \in \{1,2,3,4\}$ then the points in $\mathbb{P}$ which satisfy them is $\mathbb{P}(\mathcal{C})$*, and $\mathbb{Q}(\mathcal{C})$ is the class of quadrilaterals $\mathbf{Q} \in \mathbb{Q}$ such that $f(\mathbf{Q}) \in \mathbb{P}(\mathcal{C})$.

So for example, $\mathbb{P}(t_2 = 1)$ is the graph of orthodiagonal quadrilaterals, $\mathbb{P}(t_3 = 1)$ is the graph of equidiagonal quadrilaterals, and $\mathbb{P}(t_2 = 1) \cap \mathbb{P}(t_3 = 1) = \mathbb{P}(t_2 = 1, t_3 = 1)$ is called the **midsquare quadrilaterals** in Josefsson's terminology[1],p.81. The class of the graph $\mathbb{P}(t_1 = 1)$ is the bimajor quadrilaterals. In any case, we see that *any class whose graph is contained in one or more of $\mathbb{P}(t_i = 1)$, $i \in \{1,2,3\}$ lies in the boundary of* $\mathbb{P}$.

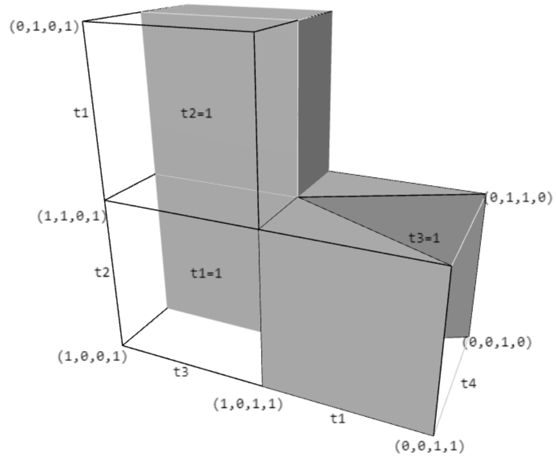


Figure 9: Graph of the boundary of $\mathbb{P}$ (in $\mathbb{P}$)

4.1. **Trapezoids.** There are two types of trapezoids in $\mathbb{Q}$, depending on whether AB and CD or BC and AD are parallel, These will be referred to these as **tall** and **short** trapezoids respectively.

You can see using alternate interior angles that $\triangle ABX$ and $\triangle CDX$ are similar for tall trapezoids and $\triangle BCX$ and $\triangle DAX$ are similar for short trapezoids. (see figure **Tall and Short trapezoids**)

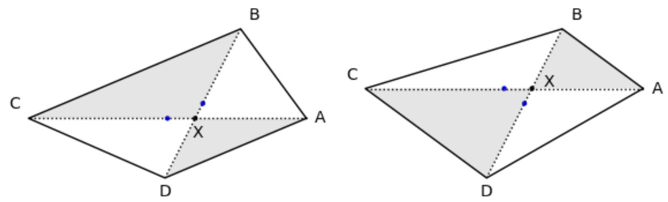


Figure 10: Short and Tall trapezoids

If $ABCD \in \mathbb{Q}$ is a tall trapezoid, then $\dfrac{|AX|}{|CX|} = \dfrac{|BX|}{|DX|}$, or $\dfrac{t_1}{2-t_1} = \dfrac{2\,t_3\,t_4}{2\,t_3\,(1-t_4)}$. This simplifies to $t_4 = \dfrac{t_1}{2}$, which gives us a description of the tall trapezoids in $\mathbb{Q}$ in terms of its parameters. Likewise, short trapezoids are characterized by the equation $t_4 = 1 - \dfrac{t_1}{2}$.

The remaining quadrilaterals (which form the vast majority) constitute 3 disjoint topological open 4-cells whose closures we will call **short, middle** and **tall quadrilaterals** respectively. This means that *in order to deform a short quadrilateral into a tall quadrilateral through an arc of quadrilaterals in* $\mathbb{Q}$*, it is necessary to pass through the middle quadrilaterals.*

We can also work out descriptions of the parallelograms, rectangles, rhombi, the square, equidiagonal quadrilaterals, and orthodiagonal quadrilaterals. From that, the dimension of each class is found and a diagram or picture of the class can be drawn in the parameter space. See the table **Algebraic description of some classes of quadrilaterals** at the end of this section.

The graphs which are 1, 2 or 3 dimensional can be visualized. For example the graph of the parallelograms, $\mathbb{P}(t_1 = 1, t_4 = .5)$ is dimension 2, in fact it is the 2-cell $\{1\} \times (0,1] \times (0,1] \times \{.5\} \subset \mathbb{P}$. It sits on the equidiagonal boundary ($t_1 = 1$) of the quadrilaterals, and is the intersection of the tall and short trapezoids, each of which is a 3-cell having the parallelograms as a common face. The 3 or 4 dimensional graphs can be visualized by looking at **cross-sections** along any one of the 4 parameters. The graphs of the equidiagonal, orthodiagonal and bimajor quadrilaterals are examples of cross-sections of $\mathbb{P}$. Another example is the graph of the biminor quadrilaterals, $\mathbb{P}(t_4 = 1/2)$. This class **separates** $\mathbb{P}$ into two disjoint open pieces.

In the figure **The space of quadrilaterals** , each point represents a 2-cell of quadrilaterals, for example, the 2-cell of parallelograms sits above the point labeled parallelograms. The cross-section of biminor quadrilaterals bisects the middle quadrilaterals into two pieces. In fact, the mapping on the interior of $\mathbb{P}$ given by $f(t_1, t_2, t_3, t_4) = (t_1, t_2, t_3, 1 - t_4)$ is a reflection, that is, a homeomorphism which is its own inverse. It maps tall quadrilaterals to short and short to tall, with fixed point set the biminor quadrilaterals.

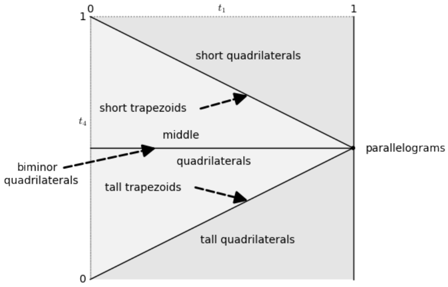


Figure 11: The space of quadrilaterals

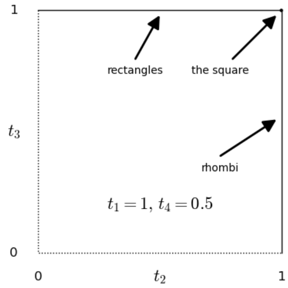


Figure 12: Parallelograms

The next table summarizes the classes of quadrilaterals, their algebraic characterizations, and their dimensions that we have discussed so far. From the table, we can see that each class of dimension n (except the square) is homeomorphic with an open n-cell with some portion of its boundary included. For example, the tall trapezoids are homeomorphic with $(0,1] \times (0,1] \times (0,1]$ and the parallelograms, the intersection of tall trapezoids with the short trapezoids, is the 2-cell $(0,1] \times (0,1]$.

Table 1: Algebraic description of some classes of quadrilaterals

Class of quadrilateral	**algebraic characterization**
tall quadrilaterals:	$t_4 \leq \frac{1}{2}\,t_1$, dim 4
short quadrilaterals:	$t_4 \geq 1 - \frac{1}{2}\,t_1$, dim 4
middle quadrilaterals:	$\frac{1}{2}\,t_1 \leq t_4 \leq 1 - \frac{1}{2}\,t_1$, dim 4
interior quadrilaterals:	$t_1, t_2, t_3, t_4 \in (0,1)$, dim 4
equidiagonal quadrilaterals:	$t_3 = 1$, dim 3
orthodiagonal quadrilaterals:	$t_2 = 1$, dim 3
bimajor quadrilaterals:	$t_1 = 1$, dim 3
biminor quadrilaterals:	$t_4 = \frac{1}{2}$, dim 3
tall trapezoids:	$t_4 = \frac{1}{2}\,t_1$, dim 3
short trapezoids:	$t_4 = 1 - \frac{1}{2}\,t_1$, dim 3
kites:	$t_2 = 1$, $t_4 = \frac{1}{2}$, dim 2
midsquare quadrilaterals:	$t_2 = 1$, $t_3 = 1$, dim 2
tall isosceles trapezoids:	$t_3 = 1$, $t_4 = \frac{1}{2}t_1$, dim 2
short isosceles trapezoids:	$t_3 = 1$, $t_4 = 1 - \frac{1}{2}t_1$, dim 2
parallelograms:	$t_1 = 1$, $t_4 = \frac{1}{2}$, dim 2
rectangles:	$t_1, t_3 = 1$, $t_4 = \frac{1}{2}$, dim 1
rhombi:	$t_1, t_2 = 1$,$t_4 = \frac{1}{2}$, dim 1
the square:	$t_1, t_2, t_3 = 1$, $t_4 = \frac{1}{2}$,dim 0

Note that in this nomenclature there are synonyms for many of the standard names of classes: parallelograms are bimajor biminor quadrilaterals, kites are orthogonal biminor quadrilaterals, rectangles are equidiagonal bimajor biminor quadrilaterals, rhombi are orthogonal bimajor biminor quadrilaterals, and the square is the only equidiagonal orthogonal bimajor biminor quadrilateral.

Next, let's examine some classes whose descriptions in the parameters are more complicated.

4.2. **Circles and quadrilaterals.** There are at least three ways a circle can interact with a quadrilateral $ABCD$: (1) It might pass through A, B, C, D, or (2) it might be tangent to all four sides of $ABCD$ in which case its interior would be in the interior of $ABCD$, or (3) it might be tangent to all four lines AB,BC,CD, and DA and in the exterior of $ABCD$. This determines three classes of quadrilaterals: *Cyclic, Tangential, and Extangential* quadrilaterals.[1]. There are algebraic descriptions in terms of t_1, t_2, t_3, and t_4, which will enable us to determine the dimension and separation properties of these and other classes of quadrilaterals. It turns out that in the tangential and extangential cases, the equations are not useful for finding specific quadrilaterals, and we introduce a new set of parameters to do that.

4.3. **Cyclic quadrilaterals.** . We know that all angles inscribed in a circle and subtended by the same chord are equal, since an inscribed angle of a circle is equal to one half of the central angle subtended by the chord. As a consequence, if $ABCD \in \mathbb{Q}$ is cyclic, $\triangle AXB$ is similar with $\triangle DXC$. So $\dfrac{t_1}{2t_3(1-t_4)} = \dfrac{2t_3t_4}{2-t_1}$. Solve this for t_1 to get $t_1 = 1 - \sqrt{1-4t_3^2t_4(1-t_4)}$, that is, $\mathbb{P}(t_1 = 1 - \sqrt{1-4t_3^2t_4(1-t_4)})$ is 3-dimensional. So the cyclic quadrilaterals form a 3-dimensional class in $\mathbb{Q}$.

Do the cyclic quadrilaterals separate the quadrilaterals? Yes, $\mathbb{Q}$ *is separated into two disjoint open sets by the cyclic quadrilaterals.* Here's a plausibility argument for this. First, for each $ABCD$ in $\mathbb{Q}$, let Circum$(ABCD)$ be the unique circle which passes through the three vertices A,B, and C of $ABCD$. Decompose $\mathbb{Q}$ into three disjoint sets: $\mathbb{C}in, \mathbb{C}, \mathbb{C}out$ where $ABCD \in \mathbb{C}in$, $\mathbb{C}$,or $\mathbb{C}out$ according to whether the vertex D lies inside, on, or outside Circum$(ABCD)$. So $\mathbb{C}$ is the class of cyclic quadrilaterals, and it is clear that if a quadrilateral from $\mathbb{C}in$ is deformed along an arc of quadrilaterals to one in $\mathcal{C}out$, then it must pass through $\mathbb{C}$.

We can visualize the graph of the cyclic quadrilaterals by drawing a typical t_2 = constant cross section of the graph. (See the drawing below). Then the graph is just the direct product of $(0, 1]$ with the cross-section. It is clear that the cyclic quadrilaterals separate the middle quadrilaterals into two disjoint open sets.

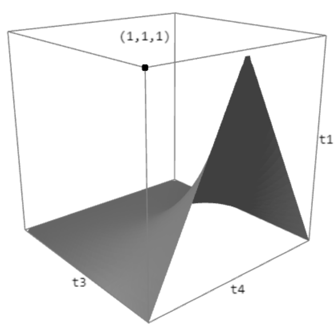


Figure 13: t_2 cross section of $\mathbb{P}$

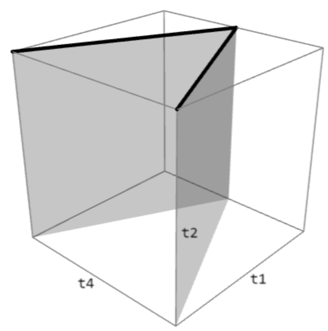


Figure 14: Graph of the isosceles trapezoids.

What about the intersection of the trapezoids with the cyclic quadrilaterals? Use a little algebra to show $\frac{t_1}{2t_3(1-t_4)} = \frac{2t_3t_4}{2-t_1}$ and ($t_1 = 2t_4$ or $t_1 = 1-2t_4$) implies $t_3 = 1$, so the intersection is the equidiagonal trapezoids, that is, the isosceles trapezoids.

Even though we can tell from the parameters of a quadrilateral if it is cyclic, we still don't know the center O or radius r of its circumcircle. But since the perpendicular bisector of any chord passes through the center, we do know that $O = (0, h)$ for some h, and from that $r = |O - A| = \sqrt{1 + h^2}$. But also $r = |O - B|$, yielding the equation $1 + h^2 = x^2 + (y - h)^2$ with $x = 1 - t_1 + 2t_3t_4c_2$ and $y = 2t_3t_4s_2$, $s_2 = \sin(t_2\pi/2)$, $c_2 = \cos(t_2\pi/2)$. Solve for h to get

$$h = \frac{x^2 + y^2 - 1}{2y} = \frac{(1-t_1)^2 + (2t_3t_4)^2 + 4(1-t_1)t_3t_4c_2 - 1}{4t_3t_4s_2}$$

.

4.4. **Tangential and extangential quadrilaterals.** The tangential quadrilaterals, the ones with an **inscribed circle**, are characterized by the simple equation $|AB| + |CD| = |BC| + |AD|$ (sums of opposite sides are equal). You can verify this equation by dropping the center of the incircle perpendicularly onto each side, and noting that each side of the equation decomposes into rearrangements of the same four numbers. The extangential quadrilaterals have a circle exterior to the quadrilateral which is tangent to all four extended sides. They were shown by Jacob Steiner in 1846 (see Wikipedia) to be characterized by the equations ($|AB|+|BC| = |CD|+|AD|$ or $|AB| + |AD| = |CB| + |CD|$). For extangential quadrilaterals in our model, this means that the exterior circle lies to the right of the major diagonal AC if $|AB| + |BC| = |CD| + |AD|$ and above the minor diagonal BD if $|AB| + |AD| = |CB| + |CD|$. We will term these **major extangential** and **minor extangential** quadrilaterals respectively.

An interesting observation about extangential quadrilaterals: Each extangential quadrilateral $ABCD$ in $\mathbb{Q}$ determines an ellipse $\frac{x^2}{(k/2)^2} + \frac{y^2}{(k/2)^2 - 1} = 1$ where $k = |AB| + |BC|$. A and C are the foci of the ellipse and B and D lie on the ellipse. When $|BD| < 2$, the quadrilateral is a major extangential quadrilateral. Not all pairs B and D yield and extangential quadrilateral. The x coordinate of B must be greater than the x coordinate of D, and B (and D) can't lie too far to the right (left): the exact ranges have not been worked out. If $|BD| > 2$, then the diagram should be reflected about the bisector of $\angle BXA$ and rescaled and the foci of the ellipse become B and D and the quadrilateral changes to a minor extangential quadrilateral.

There is a sagelet to play with the values of k and the x coordinates of B and D at *https://sagelets.cocalc.com/QuadSpace.html.*

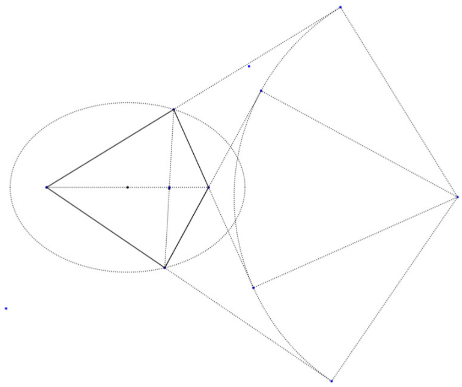


Figure 15: The ellipse of a major extangential quadrilateral

Each of the equations for the tangential and extangential quadrilaterals is a very ugly expression in the parameters t_1, t_2, t_3, t_4 equating the sum of two square roots with the sum of two other square roots. Since all four parameters are present in each equation, their graphs are not products, so we can't simply draw one cross section for each to see whether there is separation. But we do know that their graphs are 3 dimensional and we can draw their 2-dimensional cross sections in one of the parameters. Consider the $t_2 = .5$ cross sections of each equation drawn in the t_1, t_3, t_4 cube shown in the figures **View 1** and **View 2**. The cross sections for the first equation lie in the top half of the cube and converge **down** to $t_2 = 1, t_4 = 0.5$ and the cross sections for equation 2 le in the bottom half and converge **up** to $t_2 = 1, t_4 = .5$.

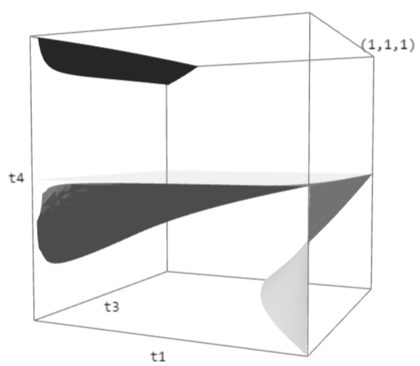


Figure 16: View 1

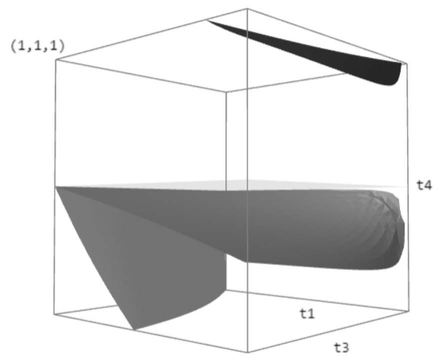


Figure 17: View 2

Do the tangential (respectively extangential) quadrilaterals separate $\mathbb{Q}$? A plausibility argument for separation analogous to the one for separation by cyclic quadrilaterals can be made here. For each $ABCD$ in $\mathbb{Q}$, let Tancir($ABCD$) be the unique circle which is tangent to the three sides DA,AB, and BC of $ABCD$. Decompose $\mathbb{Q}$ into three disjoint sets: $\mathbb{T}in$,$\mathbb{T}\ni\ltimes$,$\mathbb{T}out$ where $ABCD \in \mathbb{T}in$,$\mathbb{T}$,or $\mathbb{T}out$ according to whether the line CD meets Tancir($ABCD$) in 0, 1 , or 2 points. So $\mathbb{T}$ is the class of cyclic quadrilaterals, and it is clear that if a quadrilateral from $\mathbb{T}in$ is deformed along an arc of quadrilaterals to one in $\mathbb{T}out$, then it must pass through $\mathbb{T}$. The argument for extangential quadrilateral is analogous. The figures **View 1** and **View**

2 support these plausibility arguments. For any value of $t_2 < 1$, the graphs of the equations separate the the t_2 cross section of the t_1, t_3, t_4-cube into three disjoint open sets.

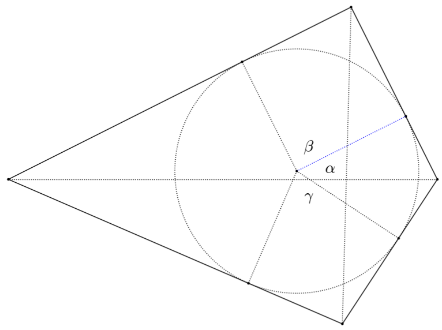


Figure 18: Tangential

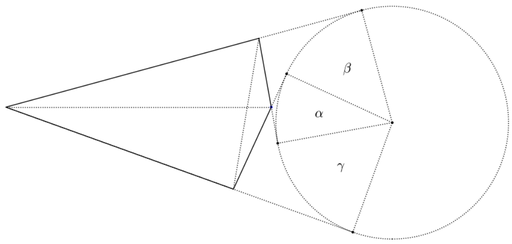


Figure 19: Major Extangential

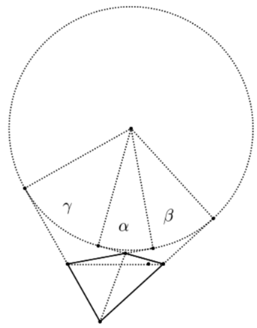


Figure 20: Minor Extangential

4.5. **The 'angle' parameters for tangential and extangential quadrilaterals.** Since we can't solve (symbolically) either defining equation for any one of the variables, it is difficult to use the equations to determine a specific tangential or ex-tangential quadrilateral. One could resort to specifying 3 values and using numerical methods to determine the value of the 4th (if it exists). This method produces unreliable results for us. However, there is another parametrization which is much less sensitive, involving the three angles α, β, and γ shown in figures 18, 19, and 20. There is a sagelet implementing this parameterization at

https://sagelets.cocalc.com/QuadSpace.html.

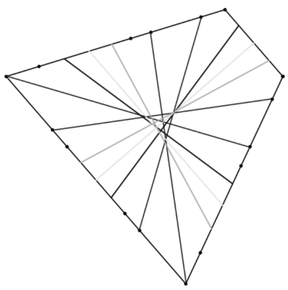
With it, you can supply α, β, γ, and see it's picture and it's location in the model.

5. Questions

This model has proved helpful to the author in gaining new insights into the study of convex quadrilaterals. There are many more questions waiting to be found and answered in the subject. Here are some that occur to me.

5.1. **Quadrisections of convex quadrilaterals.** In [4], we made the conjecture that *if P is a convex polygon with $2n+1$ vertices, then it has at most $2n+1$ quadrisections.*. In efforts to prove this, we developed formulas for calculating the area of the upper right quadrant of each possible quadrisection. (A possible quadrisection is a pair of perpendicular lines each bisecting the area of the polygon. There is one for each angle between 0 and $\pi/2$ radians.) So far those efforts have failed to prove or disprove the conjecture. We didn't make the same conjecture for quadrisections of polygons with an even number of sides because the square has the property that each possible quadrisection *is* a quadrisection.

We stumbled onto a tall isosceles trapezoid, obtained by removing an isosceles triangle from the top of the unique isosceles triangle which two quadrisections[4], with exactly 5 quadrisections (See Figure 21). It has approximate parameters $t_1 = .48735$, $t_2 = .90947$, $t_3 = 1$, and $t_4 = .24369$. Figure 22 shows the graph of the **area function**, the area of the first quadrant minus one-fourth the area of the quadrilateral, as the possible quadrisection rotates through $\frac{\pi}{2}$ radians. Since the area of any polygon changes continuously (in fact the area function is differentiable) as its vertices change smoothly, we see that there is a small positive number ϵ such that if B' and D' are within ϵ of B and D respectively, then $AB'CD'$ also has 5 quadrisections. For this example, we can take $\epsilon = .00001$.



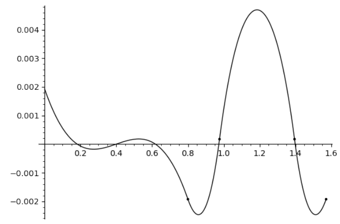
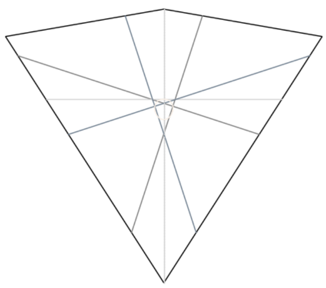


Figure 21: A trapezoid with 5 quadrisections. Figure 22: Its area function.

One can say the same about the degenerate quadrilateral $ABCD$ with $B = X$, $\angle AXD$ a right angle, and $DX = \frac{\sqrt{2}}{2}$ with parameters $t_1 = 1, t_2 = 1, t_3 = \sqrt{3}/2, t_4 = 0$, which is an equilateral triangle in the $t_4 = 0$ boundary of $\mathbb{P}$. Even though $ABCD$ is not in $\mathbb{P}$, there is a small positive number ϵ (.05 will do fine) such that each $AB'CD' \in \mathbb{P}$ with parameters $0 < t_1 - t_1' < \epsilon$, $0 < t_2 - t_2' < \epsilon$, $|t_3 - t_3'| < \epsilon$ and $0 < t_4 < \epsilon$ is in $\mathbb{Q}$ and has 3 quadrisections. Figure 23 shows an orthogonal kite with parameters $t_1 = 1$, $t_2 = 1$, $t_3 = 0.866$, and $t_4 = .05$.



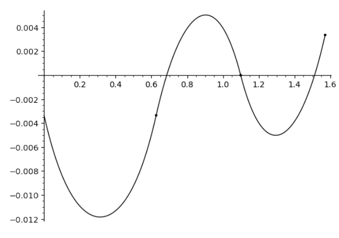


Figure 23: Kite with 3 quadrisections. Figure 24: Its area function.

However, the same is not true about the square, which has infinitely many quadrisections. Indeed, any rhombus $ABCD$ other than the square has exactly one quadrisection, as can be seen from considering Figure 25. Work out that $y = (1 - x)h$ and $s = 1/(x/h + y)$. Then the area function is $A = h/2 - (x + sy)h/2 = \frac{(1-h^2)x(1-x)}{(1-h^2)x+h^2}$. This is 0 at $x = 0$ or $x = 1$ when $0 < h < 1$ and 0 for **all** $x \in [0, 1]$ when $h = 1$ (ie when the rhombus is the square).

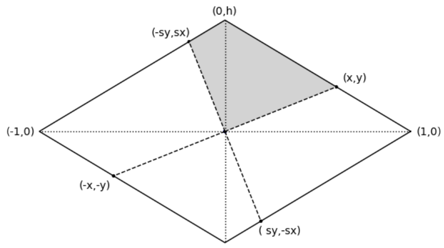


Figure 25: Nonquare rhombi have 1 quadrisection

Question 1: *Are there quadrilaterals with more than five but not infinitely many quadrisections?*

In the space of triangles, the vast majority of triangles have only one quadrisection except for a small open set about the equilateral triangle of triangles with 3 quadrisections whose boundary is an arc of triangles with 2 quadrisections.

Question 2: *Do the vast majority of quadrilaterals have only one quadrisection?*

Question 3: *Does every convex quadrilateral other than the square have only a finite number of quadrisections?*

5.2. **Compactifications of classes of quadrilaterals.** Our definition of convex quadrilateral does not include **degenerate convex quadrilaterals**, that is, convex quadrilaterals $ABCD$ which are the convex hull of a triangle or a segment. However, when one or more of the parameters t_1, t_2, t_3, t_4 is 0 or $t_4 = 1$, the resulting quadrilateral $ABCD$ **is** degenerate. So the closed 4-cube $[0, 1] \times [0, 1] \times [0, 1] \times [0, 1] = [\mathbf{0}, \mathbf{1}]^{\mathbf{4}}$ is a compact space whose boundary contains one or more copies of each degenerate quadrilateral, and two copies of each (nonsquare) equidiagonal or orthodiagonal or bimajor quadrilateral. So it's not quite the parameter space of a model for the space of convex quadrilaterals including the degenerate ones. However, if we form the **quotient space** $[\mathbf{0}, \mathbf{1}]^{\mathbf{4}}/\sim$ obtained by identifying congruent polygons, we do get a model which includes the degenerate quadrilaterals. All of the identifications occur on the boundary of $(\mathbf{0}, \mathbf{1})^{\mathbf{4}}$

Question: Describe this model. It is a 4-manifold with boundary. What does it look like? Draw the boundary.

On the way to answering the above question, it would be good to answer it for various types of quadrilaterals. For example, it is not hard to see that the compactification of the class of rectangles $\mathbb{P}(t_1 = 1, t_3 = 1, t_4 = 1/2)$ picks up just one degenerate quadrilateral $(1, 0, 1, 1/2) = f(ABCD)$ where $B = A$ and $D = C$, so the compactification of the class of rectangles is topologically a closed arc.

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